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The Reproductive Value in Distributed Optimal Control Models

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Abstract

We show that in a large class of distributed optimal control models (DOCM), where population is described by a McKendrick type equation with an endogenous number of newborns, the reproductive value of Fisher shows up as part of the shadow price of the population. Depending on the objective function, the reproductive value may be negative. Moreover, we show results of the reproductive value for changing vital rates. To motivate and demonstrate the general framework, we provide examples in health economics, epidemiology, and population biology.

Keywords

Reproductive value, distributed optimal control theory, McKendrick, shadow price, indirect effect, health economics, epidemiology, population biology

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1   Introduction

In population biology and more general mathematical demography, optimal control models for age structured systems are of great interest. For reference see chapter 8 of Grass et al. (2008) and references therein. Whenever the population is included in such distributed optimal control models (DOCM) the corresponding dynamics are usually modeled according to a McKendrick type equation (see Keyfitz and Keyfitz (1997))[1]:

$$N_a + N_t = -\mu(a,t)N(a,t) \qquad\qquad N(0,t) = B(t), N(a,0) = N_0(a). \qquad (1)$$

The state variable $N(a,t)$ represents the number of a-year old individuals at time t. The age and time specific mortality rate is denoted by $\mu(a,t)$. $N_0(a)$ describes the initial age distribution of the population and $B(t)$ equals the number of newborns at time t defined as

$$B(t) = \int_0^\omega \nu(a)N(a,t)\ da, \qquad (2)$$

where $\nu(a)$ denotes the age specific fertility rate.

Not only human populations, but also animal populations and more general age and time dynamic variables, like physical capital, can be modeled in the same framework. In the latter case, investment takes over the role of births, while depreciation constitutes the death process.

A straightforward outcome of DOCM are adjoint variables for the state variables, which can be interpreted as dynamic shadow prices.[2] They indicate the increase of the objective function (i.e. the function that has to be maximized or minimized in the optimization) if the corresponding state is increased marginally. Thus the shadow price of the population denotes the "value" of an additional individual in terms of the objective function of the model. If moreover the number of newborns is endogenous, the shadow price can be decomposed into two effects: the direct effect refers to the current situation and the indirect effect represents the forward looking component. The direct effect measures the value of the life of an additional individual and the indirect one the effect of the number of expected descendants. Interestingly the indirect effect has the

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[1] As shown in Keyfitz and Keyfitz (1997), the McKendrick equation constitutes a continuous version of the projection matrix.

[2] From now on they will be referred to as shadow prices.

same structure as the reproductive value in demography. Firstly introduced by Fisher (1930), Keyfitz (1977) presents the reproductive value at age x in the following way[3]

$$
\begin{aligned}
v(x) &= \frac{1}{e^{-rx}l(x)} \int_x^\beta e^{-ra}l(a)m(a) \ da \\
&= \int_x^\beta e^{-r(a-x)}\frac{l(a)}{l(x)}m(a) \ da,
\end{aligned}
\tag{3}
$$

where $l(a)$ denotes the probability to survive until age a, $m(a)$ the fertility rate of age a, β the oldest age of childbearing and r the discount rate, which is equal to Lotka's r. Thus $\frac{l(a)}{l(x)}$ equals the probability to survive from age x to a (conditional on being alive at age x).

Fisher regarded the birth of a child as the lending to her of a life. I.e. the newborn has a loan (of 1 unit), which can only be paid back by the birth of that child's offspring. The expected number of children over ones life equals $\int_\alpha^\beta l(a)m(a) \ da$, where α is the youngest age of childbearing. As the value of the outstanding loan has to be discounted back to age x, the term $e^{-r(a-x)}$ is included. $v(x)$ can therefore be interpreted as the discounted value (measured in units of newborns) of expected future births to a woman aged x.

In addition to showing that the reproductive value is part of the shadow price of the population in DOCM, we also derive several extensions of the classical reproductive value. More specifically our framework of the reproductive value allows for (i) changing fertility behaviour, as considered in Ediev (2007, 2009) for the case of one individual. Even endogenous fertility that depends on control and state variables is feasible within our framework. (ii) The indirect effect can also be zero or negative. Thus leaving ethic considerations aside an additional baby might be valued negatively for the population. (iii) In Fisher's formulation the discount rate was equal to Lotka's r, which is the value where the reproductive value of a newborn is exactly 1, i.e. $v(0) = 1$. In the DOCM framework the discount rate equals the discount rate of the objective function and thus reflects the patience or impatience of the population.

As already mentioned the framework we introduce is applicable for several research areas, including demography, epidemiology or biology.

As we have shown in previous work (see Kuhn et al. (2007)), the reproductive value can also be obtained in life-cycle models where age and time coincide and therefore the DOCM reduces to a standard dynamic optimal control model. For this to hold, one needs to introduce (i) the survival probability either directly as a state variable of the McKendrick type equation or alternatively as in Yaari (1956) as an exponential decay function in the objective function. Indeed both formulations are equivalent. The second requirement is to introduce (ii) the utility from (expected) children into the objective function (see e.g. Barro and Becker (1989)). Alternatively the number of

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[3]Note that we use the usual demographic notation within this part of the note and another notation (the usual notation for distributed optimal control models) later on.

expected children can be modeled as a state variable. If conditions (i) and (ii) are fulfilled, the shadow price of the survival probability can also be decomposed into a direct and an indirect effect, similar as in the case of a DOCM. However, our approach allows including interactions between age groups and and changing parameters over time (continuously or as shocks).

The rest of the paper is organized as follows. In section 2 we motivate and illustrate the mathematical framework with three examples drawn from health economics, epidemiology and from biology. Section 3 formulates a general framework DOCM that includes the McKendrick equation for the state variable of the population. Leaving optimality conditions aside (they are summarized in the appendix) we present and discuss the shadow price of population and its decomposition. Section 4 concludes.

2 Motivating Examples

In order to motivate the general framework to be developed later on, we present, within this section, models from three different areas: health economics, epidemiology and biology.

2.1 Optimal trade-off between consumption and health expenditure

Our first example is taken from Kuhn et al. (2007) and Kuhn et al. (2008), where individual life-cycle models of health behaviour (e.g. Grossman 1972, Ehrlich 2000) are extended to take into consideration not only one cohort, but the overall population (both present and future). Generally, in these models, individuals trade-off the utility from period consumption against the expected benefits from improved survival, roughly amounting to the discounted stream of future utility. Such a trade-off also occurs in models at the population level, which, however, also incorporates the benefit of future generations.

Assume that the dynamics of the population are described by the McKendrick equation (see Keyfitz and Keyfitz (1997)).

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t} \right) N(a,t) = -\mu(a,h(a,t))N(a,t) \qquad N(0,t) = B(t), N(a,0) = N_0(a), \quad (4)$$

where $N(a,t)$ denotes the number of a-year old individuals at time t and $\mu(a, h(a,t))$ the mortality rate depending on age and individual health expenditures $h(a,t)$. The initial distribution $N_0(a)$ is exogenous. The boundary condition is endogenously given by

$$N(0,t) = B(t) = \int_0^\omega \nu(a,t)N(a,t) \ da, \qquad (5)$$

where $\nu(a,t)$ denotes the exogenous time- and age-specific fertility rate. Further we introduce $A(a,t)$ as the aggregate assets for a cohort born at $t - a$. Cohort assets

4

increase with interest rate r and with the earnings $y(a)$, consumption $c(a,t)$ and net of health investments $h(a,t)$. Note that the individual's net earnings are multiplied by the current size of the age-group in order to aggregate to cohort level. Thus

$$
\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right) A(a,t) = rA(a,t) + (y(a) - c(a,t) - h(a,t))N(a,t)
$$
$$
A(0,t) = A(\omega,t) = 0 \quad \forall t
$$
$$
A(a,t) = A_0(a), A(a,T) = A_T(a) \quad \forall a, \tag{6}
$$

where $A_0(a)$ and $A_T(a)$, the initial and end distributions of age-specific assets, are both exogenous. Each cohort is assumed to hold zero assets at the time of birth and death.

The objective function is defined as the social welfare, which aggregates the discounted stream of all (present and future) individuals' per period utility from consumption, $u(c(a,t))$. Social welfare is maximized with respect to per-capita consumption and health expenditure:

$$
\max_{c(a,t),h(a,t)} \int_0^T \int_0^\omega e^{-\rho t} u(c(a,t)) N(a,t) \ da \ dt, \tag{7}
$$

where ρ is the discount rate.

Applying the theory of DOCM (see Feichtinger et al. (2003)) to the model (4), (5), (6) and (7) adjoint equation for the states and necessary first order conditions for the controls can be obtained. Solving the adjoint equation for $N(a,t)$ with the method of characteristics together with the transversality condition $\xi^N(\omega,t) = 0$ we obtain[4]

$$
\xi^N(a,t) = \int_a^\omega e^{-\rho(s-a) - \int_a^s \mu(s',h) \ ds'} \Big[u(c) + u_c(c)(y - c - h) \Big] \ ds +
$$
$$
+ \int_a^\omega e^{-\rho(s-a) - \int_a^s \mu(s',h) \ ds'} \xi^N(0, t-a+s)\nu(s, t-a+s) \ ds \tag{8}
$$

as the shadow price of population. It denotes the increase in social welfare if the population is augmented by one a-year old individual at time t.

The first integral, the direct effect, aggregates the individual's utility and it's net contribution to the cohort's wealth over its remaining life time (discounted and weighted with the survival function). This amounts to the individual's direct contribution to social welfare if it were to survive. The second integral, the indirect effect, is similar to the classical reproductive value, i.e. the discounted sum of age-specific fertility rates aggregated over the remaining life time.[5] However, there are two differences. Firstly,

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[4]From now on we omit a and t if they are not of particular importance.

[5]Ediev (2008) uses a similar expession to describe the economic-demographic potential of an individual. While he introduces this expression into a descriptive population model, we obtain the expression as the outcome of an optimization model.

the fertility rate is weighted with the shadow price of newborns $\xi^N(0, t - a + s)$, as it is their utility and not their fertility as such which contributes towards the social welfare. In the classical case, the individual's value is fixed to one. Secondly, in our formulation the fertility rate is not stable and varies over time.

For further details of the model, see Kuhn et al. (2007) and Kuhn et al. (2008).

2.2 Modeling an HIV epidemic with prevention

The second example, adds an explicit objective function to a descriptive model of an HIV/AIDS epidemic, as presented in Almeder et al. (2007). For simplicity we only consider two states within the present example instead of the four states considered in the original model. Hence, we should caution that this example probably does not reflect the epidemic in a realistic way. Nevertheless, similar results would be obtained for the full model.

The model distinguishes two groups of people, those who are susceptible, $S(a, t)$, and those who are infected, $I(a, t)$. The size of both groups diminishes with the mortality rate $\mu_S(a)$ and $\mu_I(a)$ respectively, which only depends on age.

Further, at each t some of the susceptible individuals get infected with HIV/AIDS and are therefore transferred from $S(a, t)$ to $I(a, t)$. This transfer equals $\gamma(a)\phi(u(a, t))P(a, t)S(a, t)$, where $\gamma(a)$ models the age-specific base risk of infection, $\phi(u(a, t))$ the impact of a control $u(a, t)$, leading to a reduced risk of infection, and $P(a, t)$ the age-specific rate of interaction between susceptible and infected individuals, i.e. the proportion of risky sexual contacts. The control $u(a, t)$ includes prevention programs, free condoms and in the original model also medication costs. Thus the fertility rate is influenced multiplicatively according to a function $\Theta(u(a, t)) < 1$. It is further assumed that susceptible and infected people exhibit different fertility rates $\nu_S(a)$ and $\nu_I(a)$ respectively. Furthermore we assume that a proportion α of the newborns of the infected people are not infected (entering $S(0, t)$).

$\lambda(a, a')$ represents the relative number of risky contacts between a- and a'-year old individuals. This is multiplied by the share of infected individuals in the whole age group and aggregated over all ages. The resulting $P(a, t)$ denotes the normalized prevalence.

Altogether the dynamics equal[6]

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[6] For a detailed discussion on the dynamics we refer to Almeder et al. (2007).

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right)S(a,t) = -\mu_S(a)S(a,t) - \gamma(a)\phi(u(a,t))P(a,t)S(a,t)$$
$$S(0,t) = B(t) + \alpha C(t),\ S(a,0) = S_0(a)$$
$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right)I(a,t) = -\mu_I(a)I(a,t) + \gamma(a)\phi(u(a,t))P(a,t)S(a,t)$$
$$I(0,t) = (1-\alpha)C(t),\ I(a,0) = I_0(a)$$
$$B(t) = \int_0^\omega \Theta(u(a,t))\nu_S(a,t)S(a,t)\ da$$
$$C(t) = \int_0^\omega \Theta(u(a,t))\nu_I(a,t)I(a,t)\ da$$
$$P(a,t) = \int_0^\omega \lambda(a,a')\frac{I(a',t)}{S(a',t)+I(a',t)}\ da'. \tag{9}$$

We add to the original descriptive model an objective function, amounting to the total cost of disease (present and future), which is to be minimized. Thus, we consider the discounted stream of the expenditures for controlling the infection risk (assuming a price of one for each unit of control) and the cost of disease $F(I(a,t))$, which increases in the number of infected individuals. Multiplying by -1 transforms the minimizing problem into a maximizing one, i.e.

$$\max_{u(a,t)} -\int_0^T \int_0^\omega e^{-\rho t}\Big(F(I(a,t)) + u(a,t)\Big)\ da\ dt. \tag{10}$$

Applying the theory of DOCM we can calculate the shadow prices for S and I, respectively:

$$\begin{aligned}
\xi^S(a,t) &= \int_a^\omega e^{-\int_a^s \rho+\mu_S(s')+\gamma\phi(u)P\ ds'}\left[\gamma\phi(u)P\xi^I - \int_0^\omega \zeta(t,a')\frac{\lambda I}{(S+I)^2}\ da'\right]\ ds + \\
&\quad + \int_a^\omega e^{-\int_a^s \rho+\mu_S(s')+\gamma\phi(u)P\ ds'}\xi^S(0,t-a+s)\Theta(u)\nu_S(s)\ ds \\
\xi^I(a,t) &= \int_a^\omega e^{-\int_a^s \rho+\mu_I(s')\ ds'}\left[-F_I(I) + \int_0^\omega \zeta(t,a')\frac{\lambda S}{(S+I)^2}\ da'\right]\ ds + \\
&\quad + \int_a^\omega e^{-\int_a^s \rho+\mu_I(s')\ ds'}\Big(\alpha\xi^S(0,t-a+s) + (1-\alpha)\xi^I(0,t-a+s)\Big)\Theta(u)\nu_I(s)\ ds.
\end{aligned} \tag{11}$$

We firstly discuss the shadow price, $\xi^S(a,t)$, related to susceptible individuals. The outflow into the infected group, $\gamma\phi(u)P$, is equivalent to additional mortality and therefore contributes as an additional (effective) discount rate. The direct effect (first integral) amounts to the value of an individual, ξ^I, when turning into a member of the infected group (weighted by the corresponding probability of infection) and the value a

susceptible individual has in reducing age-related interaction. The sign of the shadow price $\zeta(t, a')$ of the prevalence function $P(a, t)$ is unambiguous in the general case. For $\alpha = 0$ (newborns of infected people are always infected) $\zeta(t, a')$ is negative.

The indirect effect is similar to that of the first example and amounts to the discounted value of the individual's (expected) progeny. In this case, the impact of the control appears as a weight on fertility, taking into account its diminishing effect.

The direct effect as part of the shadow price, $\xi^I(a, t)$, of infected individuals consists of the marginal costs of an additional infected individual, as well as the (negative) value the infected value assumes due to increasing the age-interaction between risky individuals. The indirect effect is particular in the following way: While the fertility rate of infected individuals is reduced by the control, as is the case for susceptible individuals, in this case the weighted sum of the shadow price of newborn susceptibles ξ^S and that of newborn infected ξ^I is used instead due to the assumption that a proportion α of the newborns of the infectied people will be susceptible and $(1-\alpha)$ will be infected.

In this example an interesting feature of the general setting might occur. Contrary to the non-negativity of the reproductive value the indirect effect of infected people may now turn out to be negative. For positive values of α the sign is unambiguous. For $\alpha = 0$ negativity can be shown analytically. Thus all newborns of that group have a negative value in view of the objective function, as their newborns are always infected ($\alpha = 0$). This can never happen in the classical formulation of the reproductive value.

2.3 Age-structured predator-prey model

As a third example we consider a predator-prey model in an age-structured framework. A technical discussion of predator-prey models can be found e.g. in Zhao et al. (2005) or in Fister et al. (2006). In the more applied model of Gazis et al. (1973) age has been introduced into a time-dependent dynamical system by introducing three different states for the prey (e.g. calves), i.e. very young, adult and very old or sick preys. The model presented here extends age-dependency by introducing age a as a continuous variable (besides time) for both the predator and the prey. While the hunting success of the predators (e.g. wolves) is modeled similarly to Gazis et al. (1973), we employ a different model of fertility for the prey. Gazis et al. (1973) use a Verhulst equation, which excludes extreme population growth. In contrast, we follow our previous examples and use age-specific fertility rates for both species, independently of the size of the current population. However, a population-dependent fertility can easily be included. The resulting dynamics are still quite similar to the Lotka-Volterra dynamical system.

We define the number of predators R and the number of prey B as distributed states. Predators are assumed to die according to a natural mortality rate $\mu_R(a)$ and according to human hunting effectivity[7] $h_R(u(t))$, which is a function of the control variable human hunting rate $u(t)$. For the prey, the mortality rate equals the natural mortality rate $\mu_B(a)$ times the hunting efficiency of the predator population $Q(t)$, which

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[7]We assume hunting rates to be an age-independent control. The fact, that it will probably be easier to hunt young, old or sick predators can be included in the function $h_R(\cdot)$.

is an aggregated state. Here, we take into account that predators are likely to differ in their hunting efficiency according to their age; e.g. very young or very old predators may not be able to hunt at all. Finally there are also hunting activities of humans $w(t)$ (which is also a control variable) implying human hunting effectivity $h_B(w(t))$. The boundary condition $G(t)$ for the predator population is defined in similarity to the Lotka-Volterra system, where the reproductive inflow appears directly in the ordinary differential equations. Effective fertility, e.g. in the sense of surviving offspring, thus depends on the base fertility rate weighted with the aggregated state $P(t)$, an index of how easy it is for the predator population to hunt down prey to feed their young, e.g. old and young prey are probably easier to catch. The boundary condition $H(t)$ for the prey population is straightforward. Thus the dynamics of the system are

$$\text{s.t.} \quad \left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right) R(a,t) = -\mu_R(a)R(a,t) - h_R(u(t))$$

$$R(0,t) = G(t) = \int_0^\omega \nu_R(a)R(a,t)f(a)P(t) \ da, R(a,0) = R_0(a)$$

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right) B(a,t) = -\mu_B(a)B(a,t)g(a)Q(t) - h_B(w(t))$$

$$B(0,t) = H(t) = \int_0^\omega \nu_B(a)B(a,t) \ da, B(a,0) = B_0(a)$$

$$Q(t) = \int_0^\omega f(a)R(a,t) \ da$$

$$P(t) = \int_0^\omega g(a)B(a,t) \ da, \tag{12}$$

where the functions $f(a)$ and $g(a)$ denote the age-specific hunting effectivity of predators (aged a-years) and the age-specific ease of killing prey respectively.

For the objective function we assume a fairly general utility function F depending on both states (predators, prey) and both controls (hunting activities targeted at predators and prey), i.e.

$$\max_{u,w} \int_0^T \int_0^\omega e^{-\rho t} F(R(a,t), B(a,t), u(t), w(t)) \ da \ dt.$$

With this general definition it is possible to apply the model to any constellation, i.e. the animals may be pest and/or working animals and F can represent cost, utility or a combination of them.

The resulting shadow prices in their decomposition are

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$$\xi^R(a,t) = \int_a^\omega e^{-\int_a^s \rho + \mu_R(s')\ ds'} \left(F_R(\cdot) - f(s) \int_0^\omega \mu_B(s')B(t-a+s)\xi^B(s',t-a+s)g(s'') \right)\ ds +$$

$$+ \int_a^\omega e^{-\int_a^s \rho + \mu_R(s')\ ds'} \xi^R(0, t-a+s)\nu_R(s)f(s)P(t-a+s)\ ds$$

$$\xi^B(a,t) = \int_a^\omega e^{-\int_a^s \rho + \mu_B(s')Q(s''} \left(F_B(\cdot) + g(s) \int_0^\omega \nu_R(s')f(s'^R(0,t-a+s)\ ds' \right)\ ds +$$

$$+ \int_a^\omega e^{-\int_a^s \rho + \mu_B(s')Q(s''} \xi^B(0, t-a+s)\nu_B(s)\ ds. \tag{13}$$

Drawing on the discussion of our previous examples, the interpretation of both shadow prices is straightforward. However, there is one interesting additional feature. Due to the influence of the predators on the prey population and vice versa the shadow prices of the predators occur as part of the direct effect in the shadow price of the prey and vice versa. In the case ξ^R this term reflects the fact that an additional predator will reduce the number of living prey valued at ξ^B. In the case ξ^B it reflects the fact that an additional prey increases the food supply of the predators and thus increases the number of their offspring valued at ξ^R.

3   The General Model

Having considered three motivating examples we can now turn to the general framework.

The core of the model is the distributed n-dimensional state $Y(a,t) = \big(Y_1(a,t), \cdots, Y_n(a,t)\big)$, denoting the state of the system (e.g. the health of the population, assets, capital, etc.), which is described by a partial differential equation, i.e.

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right) Y(a,t) = f(a,t,N(a,t),Y(a,t),Q(t),P(a,t),u(a,t)). \tag{14}$$

$f(\cdot)$ is a function of age, time, other states $N(a,t)$, $Q(t)$, $P(a,t)$ and the control vector $u(a,t)$ (discussed in the following). For the properties of the function $f(\cdot)$ and other functions to be introduced later on we refer to Feichtinger et al. (2003) and references therein.

The initial state is given by $Y(a,0) = Y_0(a,w(a))$, where $w(a) \in W$ denotes initial age-specific controls that are fixed only once. The inflow to the system at each time (boundary condition) is modeled by $Y(0,t) = \varphi(t,B(t),Q(t),v(t))$, where $B(t)$ and $Q(t)$ denote aggregate states. $v(t) \in V$ denotes boundary controls, which are fixed at every t but independent of a.

$N(a,t)$ denotes the number of a-year old individuals at time t (or a sub-group of individuals aged a, e.g. susceptible vs. infected individuals in section 2.2) and is a distributed state of the same form as $Y_i(a,t)$. The dynamics are described by a McKendrick type equation (see Keyfitz and Keyfitz (1997)), i.e.

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right)N(a,t) = -\mu(a,t,N(a,t),Y(a,t),Q(t),P(a,t),u(a,t))N(a,t) +$$
$$+ g(a,t,N(a,t),Y(a,t),Q(t),P(a,t),u(a,t)), \qquad (15)$$

where $\mu(\cdot)$ denotes the mortality rate. In general it depends on age, time, the states and controls. By the function $g(\cdot)$ other population effects can be modeled, e.g. migration. The initial population distribution is given exogenously by $N(a,0) = N_0(a)$. The boundary condition is given by the number of newborns which is endogenously determined by

$$N(0,t) = B(t) = \int_0^\omega \nu(a,t,N(a,t),Y(a,t),Q(t),P(a,t),u(a,t))N(a,t) \ da, \qquad (16)$$

where $\nu(\cdot)$ denotes the fertility rate of an a-year old individual at time t, which may in general depend on all distributed and aggregate states, as well as on the controls $u(a,t)$. ω denotes the maximal age of an individual[8].

Remark: Note that the current setting models a one-sex population. However, this can easily be generalized by introducing states for both the female and the male population. Additionally to the female population then an interaction term $P(a,t)$ (the general expression is defined below in (18)) has to be included, capturing the interaction between females and males. However, such an extension would yield additional insight only in special cases. The analogy to the reproductive value will not change.

The number of newborns is a special form of an aggregate state, where a function depending on age, time, all states and a control is aggregated over all ages and has the same value for all cohorts alive at t (e.g. the hunting efficiency of predators, section 2.3). The general expression is

$$Q(t) = \int_0^\omega h(a,t,N(a,t),Y(a,t),Q(t),P(a,t),u(a,t)) \ da, \qquad (17)$$

where $Q(t) = \big(Q_1(t),\ldots,Q_r(t)\big)$. Finally there is also a state that models the interaction between different ages, which is important when modeling social interactions (e.g. the proportion of risky sexual contacts, section 2.2). The formal definition is

$$P(a,t) = \int_0^\omega k(a,t,a',N(a,t),Y(a,t),u(a,t)) \ da', \qquad (18)$$

where $P(a,t) = \big(P_1(a,t),\ldots,P_m(a,t)\big)$, k is a function depending on time, age, the distributed states and the distributed controls.

Finally we have to consider the control vector $u(a,t) = \big(u_1(a,t),\ldots,u_n(a,t)\big) \in U$. Together with the boundary and initial controls, $v(t)$ and $w(a)$, previously introduced,

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[8]Note that this is no restriction to the model, since ω can be set arbitrarily large, e.g. to 200 years for a human population.

it allows to influence the dynamical system. The objective is to maximize the outcome, which is defined by an objective function, i.e.

$$\int_0^T \int_0^\omega e^{-\rho t} L(a, t, N(a,t), Y(a,t), Q(t), P(a,t), u(a,t), v(t), w(a)) \ da \ dt +$$

$$+ \int_0^\omega e^{-\rho T} l(a, Y(a,T)) \ da, \tag{19}$$

where $L(\cdot)$ is a function of all states and all controls (representing e.g. costs, social welfare, etc.). The discounted values (ρ denotes the discout rate) are aggregated over age and time. Additionally the last integral denotes the salvage value, the discounted weight of the states at the end of the time horizon (e.g. assets). Whereas in the above form the salvage value only depends on $Y(a,t)$, it also depends on the population $N(a,t)$ in general. The result would only change for cohorts that are born at $t = T - \omega$ or later.

The whole general DOCM reads as follows (time and age arguments are skipped)

$$\max_{u,v,w} \int_0^T \int_0^\omega e^{-\rho t} L(a, t, N, Y, Q, P, u, v, w) \ da \ dt + \int_0^\omega e^{-\rho T} l(a, Y(a,t)) \ da$$

$$\text{s.t.} \quad \left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right) N(a,t) = -\mu(a, t, N, Y, Q, P, u) N + g(a, t, N, Y, Q, P, u)$$

$$N(0,t) = B(t) = \int_0^\omega \nu(a, t, N, Y, Q, P, u) N \ da, N(a,0) = N_0(a)$$

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right) Y(a,t) = f(a, t, N, Y, Q, P, u)$$

$$Y(0,t) = \varphi(t, B, Q, v), Y(a,0) = Y_0(a,w)$$

$$Q(t) = \int_0^\omega h(a, t, N, Y, Q, P, u) \ da$$

$$P(a,t) = \int_0^\omega k(a, t, a', N, Y, u) \ da'$$

$$u \in U, \ v \in V, \ w \in W. \tag{20}$$

By applying the theory of DOCM (see Feichtinger et al. (2003)) it is possible to derive necessary optimality conditions as well as a system for the adjoint variables for this problem.

Those conditions which are not immediately relevant for our main argument are shifted to the appendix. Thus, we only formulate the dynamics of the adjoint variable for the population $\xi^N(a,t)$, i.e.

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right) \xi^N(a,t) = (\rho + \mu + \mu_N N - g_N)\xi^N - L_N - \xi^Y f_N -$$

$$- \xi^N(0, t - a + s)(\nu_N N + \nu) - \eta^Q h_N - \int_0^\omega \zeta P_N \ da', \tag{21}$$

where $\xi^Y(a,t)$, $\eta^Q(t)$ and $\zeta(a,t)$ are the adjoint variables of Y, Q and P respectively. All adjoint variables can be interpreted as dynamic shadow prices, i.e. they indicate the increase of the objective function if the corresponding state is increased marginally. E.g. $\xi^N(a,t)$ denotes the increase of the objective function if the population is increased marginally at age a at time t (or by one a-aged individual at t if the population is large enough). The term shadow price has already been used in the examples.

Together with the transversality condition $\xi^N(\omega,t) = 0$ the shadow price of the population can be solved with the method of characteristics for all cohorts whose maximal life horizon ends before the planning horizon T

$$\xi^N(a,t) \;\; = \;\; \int_a^\omega e^{-\int_a^s \rho + \mu + \mu_N N - g_N \; ds'} \left(L_N + \xi^Y f_N + \eta^Q h_N + \int_0^\omega \zeta P_N \; da' \right) \; ds +$$

$$+ \int_a^\omega e^{-\int_a^s \rho + \mu + \mu_N N - g_N \; ds'} \xi^N(0, t - a + s)(\nu + \nu_N N) \; ds. \qquad (22)$$

For all cohorts that are born later, i.e. in the interval $[T - \omega, T]$ the upper bound of the integral is $T - (t - a)$. In the following discussion we will not deal with this special (and slightly different) case, which we have also omitted from discusssion in the context of our examples.

The first integral sums up the (marginal) contribution to the objective function by the additional individual and its effect on the states (terms in the brackets). The discount factor $e^{-\rho(s-a)}$ is augmented by the conditional survival probability $e^{-\int_a^s \mu \; ds'}$ (given to be alive at t) and two additional terms accounting for the population level. The first term $e^{-\int_a^s \mu_N N \; ds'}$ reflects the density dependence of the mortality. The second one $e^{\int_a^s g_N \; ds'}$ reflects the density dependence of other endogenous population changes.

Thus the first integral, called the *direct effect*, equals the net contribution to the optimized value of social welfare which is directly related to the own life of an additional individual.

The second integral aggregates the contribution by the expected descendants of the additional individual. It is the product of two factors over the inidvidual's remaining life span discounted by the same factor as the direct effect.

The first one is the sum of the fertility rate and the change (positive or negative) of the fertility rate due to the higher number of a-year old individuals. This is a little more general than in Fisher's reproductive value, where no change in the fertility rate is considered. The second one is the shadow price of a newborn. As an additional newborn is added it is valued in a way similar to the parent namely by the shadow price. In Fisher's case this value equals 1, because of two reasons. (i) As already mentioned in the introduction Fisher regarded the birth of a child as the lending to him of a life, i.e. a loan of 1 unit. Thus each new child has exactly value 1. In our case the value of the system is expressed in terms of the value of the objective function (e.g. if the objective function measures utility, the value of individuals is expressed in units of utility). Thus also the value of an additional individual and a newborn (which is the

shadow price) is measured in units of the objective function. (ii) The discount rate r in Fisher's reproductive value equals Lotka's r, which is defined as[9]

$$\int_0^\omega e^{-rs - \int_0^s \mu(\cdot)\ ds'} \nu(s, \cdot)\ ds = 1. \tag{23}$$

This is not valid in our case, as we use a subjective discount rate, which reflects the impatience (in economic models where capital or assets are included this factor often equals the market interest rate for simplicity). Moreover our so-called *indirect effect* is more flexible, as we also allow for changing fertility rates. Overall, the indirect effect seems to be a more general expression of Fisher's reproductive value since it constitutes "natural" outcome of an DOCM with population and endogenous number of newborns.

Due to the definition the classical reproductive value is always non-negative, i.e. positive before and within the fertile period and zero afterwards. However, in the DOCM framework this may fail. It is also possible and in some cases plausible that the indirect effect is negative (before and in the fertile period - afterwards it is zero anyway).

An indirect effect that is zero for all ages of one cohort can only occur (ignoring the case of an overall zero shadow price) if the fertility rate depends on the population (i.e. $\frac{\partial \nu}{\partial N} \neq 0$ for at least one (a, t) and corresponding states) and if the elasticity between them equals -1, i.e.

$$\epsilon(\nu, N) := \frac{\partial \nu}{\partial N} \frac{N}{\nu} = -1, \tag{24}$$

which means a 1-percent increase (decrease) in the population aged a at t implies a 1-percent decrease (increase) in the fertility rate. For the DOCM this means that an increase (decrease) in the population has no effect on the objective function through an according change in the number of expected descendants (at the corresponding state values). The possible increase (depending on the sign of the shadow price) in the objective function is compensated by a decrease of the fertility rate. Interestingly the above elasticity results if the expected number of newborns of a-year old individuals at t is maximized with respect to the population aged a at t, in a static way, i.e. without considering any intertemporal effects: $\max_{N(a,t)} \nu(N(a,t), \cdot)N(a,t)$. The necessary first order condition yields $\epsilon(\nu, N) = -1$. If $\epsilon(\nu, N) < -1$, the number of the expected births to a-aged individuals would be greater if the population of that age were smaller. Assuming that the optimal N (optimal in view of the static optimization problem) can be reached in the DOCM by chosing the optimal controls adequately, this means that the population should grow as fast as possible. For $\epsilon(\nu, N) > -1$ the interpretation is the other way around.

Static maximization with regard to the number of newborns for every age group implies the maximization of the population as a whole, which is reached exactly at the carrying capacity. Therefore, the indirect effect has to be zero in this case. An

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[9]Recalling (3) this equation equals $\int_0^\infty e^{-rx} l(x) m(x)\ dx$ in demographic notation.

analogous result for the reproductive value has been shown in the descriptive model of Samuelson (1977). For animal populations it is known that fertility depends on population density. A recent paper by Lutz et al. (2006) provides empirical evidence that for many countries the same is true for human populations. However, up to our knowledge there are no studies so far that consider this effect.

4   Conclusions

The aim of the paper is to show that Fisher's reproductive value results from the first order conditions in a large class of distributed optimal control models. A sufficient condition for our result to hold is the existence of a population state which evolves according to a McKendrick type equation together with an endogenous number of newborns. The other state dynamics and aggregate functions, as well as the objective function, can be of any form.

Our main result is the decomposition of the shadow price of population - which denotes the change in the objective function if the population is increased marginally - into a direct and an indirect effect. The latter one denotes the additional value of the expected descendants and seems to be a more general version of Fisher's reproductive value. Contrary to the reproductive value concept so far, our expression for a generalized reproductive value can be negative as well and also allows for changing fertility rates. We have chosen three exampels from health economics, epidemics and biology to motivate the presented framework and results.

The novelty in our approach is the fact that we obtain the reproductive value as the result of a normative model. It would be very interesting to go into the question, as to whether there exist special conditions under which the indirect effect is negative. Further it remains an open question if the presented results are also valid for a class of the optimal control models of heterogenous systems, which include DOCM as special case.

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A   Appendix

The distributed, initial and boundary Hamiltonian of the general model reads as follows

$$
\begin{aligned}
\mathcal{H}(\cdot) &= L + \xi^N(-\mu N + g) + \xi^Y f + \eta^B \nu N + \eta^Q h + \int_0^\omega \zeta k(a, t, a', u) \ da' \\
\mathcal{H}_0(\cdot) &= \xi^N(a, 0) N_0(a) + \xi^Y(a, 0) Y_0(a, w) + \int_0^T L(a, t, w) \ dt \\
\mathcal{H}_b(\cdot) &= \xi^N(0, t) B(t) + \xi^Y(0, t) \varphi(t, v) + \int_0^\omega L(a, t, v) \ da.
\end{aligned} \tag{25}
$$

Applying distributed optimal control theory (see Feichtinger et al. (2003)) we obtain the following adjoint system

$$
\begin{aligned}
\xi_a^N + \xi_t^N &= (\rho + \mu + \mu_N N)\xi^N - L_N - \xi^N g_N - \xi^Y f_N - \eta^B(\nu_N N + \nu) - \eta^Q h_N - \int_0^\omega \zeta^P k_N \ da' \\
\xi_a^Y + \xi_t^Y &= (\rho + f_Y)\xi^Y - L_Y + \xi^N \mu_Y N - \xi^N g_Y - \eta^B \nu_Y N - \eta^Q h_Y - \int_0^\omega \zeta^P k_Y \ da' \\
\eta^B &= \xi^N(0, t) + \xi^Y(0, t)\varphi_B \\
\eta^Q &= \xi^Y(0, t)\varphi_Q + \int_0^\omega L_Q + \xi^N(-\mu_Q N + g_Q) + \xi^Y f_Q + \eta^B \nu_Q N + \eta^Q h_Q \ da \\
\zeta^P &= L_P + \xi^N(-\mu_P N + g_P) + \xi^Y f_P + \eta^B \nu_P N + \eta^Q h_P,
\end{aligned} \tag{26}
$$

together with the transversality conditions[10]

$$
\begin{aligned}
\xi^N(a, T) &= 0 & \xi^N(\omega, t) &= 0 \\
\xi^Y(a, T) &= l_Y(a, T) & \xi^Y(\omega, t) &= 0.
\end{aligned} \tag{27}
$$

Finally the necessary first order conditions can be derived from

---

[10]Note that we assume here no salvage value of population. However, this is no restriction to the model, since ω can be chosen arbitrarily large, e.g. 200 years.

$$
\begin{aligned}
\mathcal{H}(a,t,u^*(a,t)) &\geq \mathcal{H}(a,t,u(a,t)) && \forall\, u(a,t) \in U \\
\frac{\partial \mathcal{H}_0}{\partial w}(a_0, w^*(a_0))(w - w^*(a_0)) &\leq 0 && \forall\, w \in W \\
\frac{\partial \mathcal{H}_0}{\partial v}(t_0, v^*(t_0))(v - v^*(t_0)) &\leq 0 && \forall\, v \in V,
\end{aligned}
\tag{28}
$$

where $u^*(a,t)$ denotes the distributed, $w^*(a_0)$ the initial and the $v^*(t_0)$ the boundary optimal control.

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