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# *Extrapolative Projections of Mortality: Towards a More Consistent Method Part I: The Central Scenario*

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## Abstract

After a comparative study of the Lee-Carter forecasting method and looking into the direct extrapolation of mortality by age and sex, this paper advocates the use of the latter method. The method is, however, supplemented by additional procedures in order to improve its efficiency in the short run and preclude implausible mortality patterns in the long run. The short-run efficiency is improved by building the forecast on data from the most recent periods of age/sex-specific duration, when the mortality dynamics exhibit a steady trend. In the long run, the rates of the decline in mortality are assumed to converge to a plausible function of age and sex, which is derived from the data based on the assumption that it is a monotonic function of age. The framework proposed also provides a natural basis for developing multi-regional projection methods and also for introducing uncertainty into the projection.

## Keywords

Mortality forecasting, direct extrapolation, age-specific death rates, Lee-Carter method.

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# Extrapolative Projections of Mortality: Towards a More Consistent Method.
# Part I: The Central Scenario

Dalkhat M. Ediev

## 1. Introduction

Both the strengths and weaknesses of extrapolative methods for projecting mortality lie in the simplicity of the method. Avoiding to analyze casual factors underlying the observed dynamics may turn the task of projection into a relatively easy and very practical routine that is relatively free from subjective judgements. Ignorance of these factors, however, may in turn downgrade the relevance of the extrapolation, especially in the long run.

> Extrapolation is too often imposes a *fixed* rate of change based on the history of the time series, so this method will clearly err in forecasting phenomena experiencing turning points or changes in rates of decline. (McNown and Rogers 1989: 646).

In projecting life expectancy, e.g., it is often considered implausible to assume the unchecked continuation of past improvements into the future, because improvements in the past were mainly due to a decline in mortality at young ages, while recent trends as well as those expected for the future, are attributed to a decline in mortality at older ages in most modern populations (e.g., Wilmoth 2005). Relative to the decline in mortality at young ages, the decline at old ages has a more moderate effect on life expectancy at birth (Keyfitz 1985, Wilmoth 1998) and might be limited if there are biological limits to human longevity. The dispute on these limits and on future growth in life expectancy is not yet resolved, however (e.g., Olshansky et al. 2001; Oeppen and Vaupel 2002; De Grey 2006).

One way of addressing the anticipated change in the trend that is due to changes in underlying factors might be to correct the extrapolation model. Instead of linearly extrapolating life expectancy, e.g., it was proposed to project it by a logistic curve or otherwise in such a way that year-to-year improvements in life expectancy decrease as life expectancy grows (Coale 1981; Bulatao and Bos 1989; Trewin 2000; UN Population Division 2005).

Another practical way that this paper is focused on is to disaggregate and extrapolate mortality by age, sex and, possibly, other relevant variables. Although individual mortality rates may also experience turning points, the approach is likely to be more consistent as it explicitly reflects the dynamics of mortality rates at different age groups. After proper transformation, individual age/sex-specific mortality rates do show consistent trends, which can be extrapolated rather robustly. The approach has been used in many applications and model frameworks. The widely used method by Lee and Carter (1992a; the method was also proposed by Gómez de León (1990)) as well as methods based on extrapolating parameters of mortality models (e.g., McNown and Rogers 1989; Lutz et al. 1997) or on shifting the age profile of mortality rates (Bongaarts and Feeney 2003; Bongaarts 2005) also represent an approach to mortality projection by age/sex groups.

Further decomposition of mortality, e.g., by causes of death, has not found enough justification as it leads to an unjustified complication of the method and makes projection assumptions less transparent while potentially resulting in internal inconsistencies with respect to interrelations between several components of mortality (Wilmoth 2005).

Limitations of projecting mortality decomposed into age/sex-specific rates are linked to a possible lack of correspondence between *projected* individual trends. Assuming, for example, mortality to decline at every age by a constant rate observed from the past is likely to result in erratic developments in age structure of mortality, including crossovers of projected age-specific mortality trends. In such cases, special arrangements are to be made so as to avoid inconsistent developments in projected trends. The US Census Bureau, for example, once used the following conditions imposed on projected rates:

> 1. No 2050 death rate was allowed to be higher than it was in 1994
> 2. No male rate was allowed to ever be lower than the equivalent female rate
> 3. Within a given race-sex group, the death rates must steadily rise from age 25-29 to 100+
> 4. No death rate was permitted to improve more than 3 percent per year during the 1994 to 2050 period.
>
> (Day and US Census Bureau 1996).

In another case, annual rates of mortality decline by age and sex were projected to converge to a target value for 2039 and were assumed to be identical afterwards (Gallop 2007). In still other examples, the projected age-specific rates of mortality decline were kept different and yet, taken from expert judgement rather than directly from past observations (Hollmann et al. 2000; Wilmoth 2005).

The method by Lee and Carter is subject to the same problems of inconsistent mortality profiles emerging from projecting past trends into the future (e.g., Lee and Miller 2001). Indeed, this was noted already at the moment of publication of their influential paper:

> From this analysis it becomes clear that forecasts of mortality identical to Lee and Carter's will be produced by directly projecting each age-specific mortality rate at its own historical rate of exponential decline. … One concern is that if each age-specific rate is allowed to change at its own individual rate, the projected age profile of mortality may deviate from plausible, historically observed patterns. (McNown 1992)

See also a similar note by Alho (1992) in the same journal. Addressing these concerns, however, the authors of the method pointed to several distinctions between their method and that of directly extrapolating age-specific rates (Lee and Carter 1992b). See also some notes on dissimilarities between the methods further down in the paper.

On many occasions, scholars have expressed concerns about plausibility of 'blind' extrapolation, when its results seemed to violate their expectations, e.g.:

> … the temptation to extrapolate (for instance the expectation of life or of active life) has to be firmly rejected. Among other reasons this is because mortality at younger ages has been falling to the point where it now cannot fall much further, while at older ages mortality has been resistant to medical and sanitary advance. (Keyfitz 1993)

Despite this sceptical concern of one of the most experienced demographers, however, the mortality rate at age 0 averaged over populations included in the HMD (Human Mortality Database) has fallen by 30% in the decade starting from 1991, which was even slightly higher than in the preceding decade (27%), while average mortality rate at age 95 has fallen by about 8% in both decades. Quite notably, the overall decline of HMD-averaged mortality was only slightly more moderate in the 1960s compared to the 1990s and had very similar age pattern.

This example—as non-representative as it may be—points to the fact that properly conducted, extrapolation may still be a prominent tool of forecasting mortality phenomena, as long as they show a sustained trend in the past. Indeed, it seems to be a common method for official projections (e.g., Social Security Administration 1987; Day and US Census Bureau 1996; Wilmoth 2005; Gallop 2007), which may also be attributed to the fact that extrapolation is often considered—though in many instances wrongly so—to be free from subjective judgements.

The purpose of this paper is to improve the performance and consistency of the most straightforward method of directly extrapolating mortality rates by age and sex. We start by a comparison of the Lee-Carter model with direct extrapolation of mortality rates, as the improvements proposed may equally be developed within both models (Ediev 2007). Based on the comparative study, the direct extrapolation is chosen as the more effective tool and is supplemented by consistency adjustment procedures and other improvements in the final parts of the paper. More details on the Lee-Carter model may be found elsewhere (e.g., Gómez de León 1990; Lee and Carter 1992a; Wilmoth 1993; Lee and Miller 2001; Booth et al. 2002; Girosi and King 2006), see also an extensive review by Booth (2006).

We do not consider here another important approach to extrapolation, which is based on fitting a parametric model to the mortality age profile and projecting the parameters of the model (e.g., McNown and Rogers 1989; Lutz et al. 1997). That method is convenient in keeping the projected mortality profiles consistent. However, its relevance to age profiles emerging in the future may not be automatically granted. Neither do we consider the shifting model (Bongaarts 2005), which assumes the age function of mortality to be shifted upwards along the age scale and extrapolates the speed of the shift, which may be considered as a rough approximation to actual mortality dynamics but cannot be applied to modelling mortality at young ages.

## 2. The Lee-Carter Model vs. Direct Extrapolation: A Comparative Study of Autocorrelations

The Lee and Carter's (LC) model may be summarized by the following expression for log-mortality rates:

$$\ln\left[m(x,t)\right] = a_x + b_x k_t + \varepsilon_{x,t}, \qquad (1)$$

here $m(x,t)$ is the central death rate for age $x$ at time $t$; $a_x$, $b_x$, $k_t$ are the model parameters which represent, consequently, overall age dependency of the mortality; the relative age-specific rates of mortality change; and overall mortality level at time $t$; and

$\varepsilon_{x,t}$ is the error term. Hereinafter, we explicitly consider only the age variable, although all the models considered may also include sex, race and other variables, which might be of interest in practice.

By comparison, direct (linear) extrapolation of log-mortality rates would imply the following model:

$$\ln\left[m(x,t)\right] = a_x + b_x t + \varepsilon_{x,t}. \quad (2)$$

It is obvious that the two models are essentially identical when the indicator of the overall mortality level $k_t$ is modelled as a *linear* function of time. Indeed, this is the usual case for applications to *deterministic* mortality forecasting. Despite this major similarity, however, there are important differences between the methods, which deserve closer attention. We classify these differences into two groups: those related to parameter *estimation* and those related to mortality *forecasting*.

### 2.1. Estimation of Past Trends

Estimation of parameters in the linear extrapolation model (2) is straightforward, e.g., based on ordinary least squares procedures. (For a comparative study of alternative approaches one may refer to (Wilmoth 2005).) Indeed, it is reasonable to fit linear regressions only to data that show a linear trend. Otherwise, one may either consider nonlinear regressions or fit the linear model only to a most recent portion of the data, which shows a linear trend. Hence, direct extrapolation of age-specific rates may, in principle, be based either on different models (linear, polynomial, logistic, etc.) or on data periods of different length at different ages. Along with this flexibility, however, the method may potentially lack statistical efficiency, from ignoring old data, which are not consistent with the most recent linear trends, and also from neglecting possible cross-age correlations in residuals.

The LC model is more efficient in addressing these two situations as it assumes, *firstly*, correlations between age-specific deviations from the linear trend. Were these correlations significant, the method could even provide statistically more efficient estimates for parameters. *Secondly*, the model does not imply a linear trend to be observed in data period and, therefore, may be based on observations from longer time periods. (In *projecting* the overall mortality index $k_t$, however, one may again make use of the most recent linear trend; also see (Booth et al. 2002) for using the method based on the most recent data period only.) With respect to the second argument it is important that age-specific trends are assumed to be correlated (in the sense that their departures from linearity are correlated). Otherwise, there might be no added value from estimating together many unrelated trends. Hence, the potential superiority of the LC model compared to the direct extrapolation of rates nests in correlations between age-specific trends in the aforementioned sense. In the absence of such correlations, the method might be less efficient as compared to direct extrapolation as it does not provide a similar flexibility with regard to choosing age-specific durations of the data period and also with regards to selecting different models at different ages.

Hence, it is worth to examine whether such correlations exist in the data and how significant they are. We address this task by examining empirical correlations of three sorts. *First*, we examine cross-age correlations between residuals of the linear model (2). Ideally for the LC model, these residuals must have correlation coefficients close to unity. Otherwise, statistical supremacy of the method over separate consideration of age-specific mortality trends might be doubtful. *Second*, we examine empirical correlations between residuals of the LC model itself, as—given that the model provides a significant gain in statistical efficiency—these correlations should be significantly reduced as compared to the other two correlations computed. *Third,* correlations are calculated in order to control for possible effects of nonlinear trends because correlations of residuals from the linear trend might also be significant if some *nonlinear* model is relevant to the overall trend in mortality. Hence, one could also test for correlations between residuals of properly chosen age-specific nonlinear regressions. We use a somewhat simplified approach, testing for correlations between age-specific rates of mortality decline. Since mortality rates usually show a high degree of short-term volatility, the variation of the age-specific rates of mortality decline may only to a smaller extent be subject to the effect of relatively slow non-linear developments in the overall trends. Hence, high correlations between age-specific rates of mortality decline would be a good argument in favour of the LC model against direct nonlinear extrapolations of the age-specific rates. Putting it differently, low correlations between age-specific rates of mortality decline imply that cross-age correlations, if any, potentially captured by the LC model are of minor importance compared to the independent variation of errors.

Results of calculating the first two correlations mentioned for female populations of six countries (USA, Japan, France, Russia, Sweden, and Austria)[1] for the period 1980 to 2004 are presented in Figure 1. (Results for male populations—not shown—are similar.) Light areas in figures correspond to low correlation; also note that negative correlations are marked by grid lines. The relatively short period of 1980-2004 was chosen in order to limit the effects of nonlinear trends, which are more likely to be observed in a longer data (results for long periods of data are coming further down). Correlations between the residuals of the *linear model* (upper-right triangles in Figure 1) are almost entirely positive for Russia; correlations for USA, Japan, and France consist of continuous areas of positive and negative correlations; and for Sweden and Austria the data show an erratic pattern of insignificant correlations varying around zero (which probably reflects a smaller population size of these two countries). Even in the Russian case, however, the correlations are relatively high for only a subset of age groups. High correlations for Russian mortality at ages 20-65 reflect the well-known anomalies of Russian mortality at young adult ages reduced first by Gorbachev's anti-alcohol campaign in the 1980s and then aggravated by the economic decline during the 1990s (e.g., Shkolnikov et al. 1996, Mesle and Vallin 2002), which resulted in deviations from a linear trend in 1980-2004. Apparently, the pattern of correlations of residuals of the linear model—even in the case of Russia—does not provide a strong support for the Lee-Carter model. This point is further strengthened by patterns of correlations of residuals of *the LC model* itself (lower-left triangles in Figure 1). (Note that we used a simpler method to estimate parameters of the model (Ediev 2007),

[1] Data used in the paper—unless otherwise stated—are taken from the *Human Mortality Database* sponsored by University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany), www.mortality.org or www.humanmortality.de.

without using the SVD procedure; the method is close to the one proposed by Lee and Carter (1992a) and provides results similar to those based on SVD). The LC model does not provide a significant reduction of correlations, except for the case of Russia. In the Russian case, however, the reduction of correlations in residuals at young adult ages brought about other—artificial—correlations at child ages. The reason is that the dynamics of Russian mortality showed essential differences at child ages, at young adult ages, and at old ages. These were differences both in the timing of changes and even in the *direction* of changes of mortality (infant mortality, e.g., continued to decrease even through the 1990s, despite an overall mortality increase in Russia). Such complex mortality dynamics do not fit the very framework of the LC model.

Figure 2 presents correlations (for the same six countries) for a longer period, i.e., 1960 to 2004. For data from a longer period of time, linear trends are, indeed, less relevant, as is reflected in somewhat more pronounced correlations for all countries, except maybe for Russia, in which case a mixture of different correlation patterns in the periods prior to and after the Gorbachev reforms made the picture less clear. The fact that correlations between the residuals of the linear model—even as moderate as they may be—are due to nonlinear trends expressed by the data rather than to correlated deviations from trends, is illustrated by nearly negligible correlations between age-specific *rates of mortality decline*, which are presented in Figure 3 for the period 1980-2004. In the same figure, autocorrelations for the LC model are replicated to facilitate a comparative study. Correlations of the residuals of the LC model are much more pronounced compared to those of rates of mortality decline, which implies that the LC model was not able to capture non-linear tends for all ages. Instead it has captured only the overall trend of the average log-mortality rate, hence neglecting different developments of the trend at different ages. Even Russian data, clearly affected to several period shocks in 1980-2004, which should increase the correlations of the rates of mortality decline, do not contain correlations high enough to support a model of collinear mortality trends at different ages of the LC type.

Figure 4 presents results on even longer data periods for males as well as for females. Correlation patterns of residuals of linear trends and of the LC model for female and male populations of three countries (US, 1933-2004; France 1899-2004; and Sweden, 1806-2004) are shown in the figure. As might be expected, residuals of linear trends show stronger correlations for longer data periods, as they are more likely to have non-linear developments in long run. High cross-age correlations among residuals for long data periods may additionally be explained by a higher correlation of short-term fluctuations in the period before the modern epidemiological transition, due to more pronounced effects of external factors affecting mortality at all ages (e.g., famines or epidemics). Still another possibility for enhanced autocorrelation with historical estimates lies in certain limitations of data quality and of reconstructing techniques (age heaping and methods used to eliminate it, e.g., could artificially enhance autocorrelations). These considerations are illustrated in Fig. 5, where autocorrelations are presented for three countries (Austria, England and Wales, and Sweden) with available data and reconstructions for the 19th century (Ediev and Gisser 2007, Human Mortality Database). Notably higher autocorrelations in the case of Austria are due to smoothing procedures used in reconstructing historical life tables (Ediev and Gisser 2007). Similarly, some artificial

regular patterns in the case of England and Wales seem to be due to limitations of reconstruction techniques.

Even in the case of extremely long data periods, however, correlations in residuals are not uniform and only reflect the similarities in mortality dynamics for subsets of age scale rather than supporting the concept of universal affinity in mortality trends. In such a case, implying the LC model results only in distorting the correlation patterns rather than in reducing them. The male population of the US and the female population of France illustrate an interesting case, when a population can be divided into two groups (those younger than, approximately, 45 years of age and those older), which are internally correlated in mortality dynamics and, at the same time, are not correlated with each other. Fitting the LC to these populations produced a model, which is describing more or less well the mortality dynamics at young ages (this is seen from the correlations of the residuals being reduced at these ages) while ignoring the correlations at old ages and introducing artificial negative correlations in the mortality dynamics between young and old ages. The dynamics of mortality of the male French population were even more complex, presumably because of the two world wars which, in addition to the overall divergent trends, influenced in a distinctive manner the mortality dynamics at ages involved in military service. For the same populations, correlations among age-specific rates of mortality decline (not shown in the figure) were quite low, which suggests that the mortality dynamics would better be described by some non-linear trends with correlation patterns more complex than suggested by the LC model.

In summary, the correlation patterns presented are in favour of separately estimating the age-specific mortality trends. There are indeed some correlations in residuals from linear or non-linear trends. In principle, these correlation patterns might be used in some simultaneous estimation procedures. However, it is to be doubted whether such a complication might really result in more efficient estimates. The main reason behind the doubt here is that correlation patterns are not fixed for all times, nor for all populations. Matrices of correlations between residuals are very dynamic and change considerably from period to period and from population to population. Probably a good interpretation for this type of dynamics would be that age-specific mortality rates follow some separate paths which are not strictly linked to each other; yet, these paths are linked by some loose relations, which—in statistical terms—are expressed differently at different periods and for different populations. (One may also note that cross-time autocorrelations may significantly reduce the efficiency of estimating cross-age correlations from relatively short data periods, which are usually available.) In view of such dynamics, it might be a better approach to estimate age-specific trends separately, without taking note of possible (varying) correlations between them. That will provide the aforementioned flexibility in choosing age-specific durations of the data period, as well as the model type itself. At the same time, when it comes to projecting mortality trends, special arrangements are to be made to keep projected trends at different ages linked to each other, so that inconsistent divergences arising from ignoring the aforementioned ‘loose correlations’ will be precluded (we discuss such consistency-reinstating arrangements later on in the paper).

One may also take a different perspective to compare estimation of mortality developments by directly fitting age-specific trends and by using the LC method. The latter

method assumes time to be *'calculated' from data*. In practice, however, the *timing* of mortality dynamics may be quite different for different ages. Russian data present an interesting example. In the mid-1960s to the mid-1980s, the average of age-specific log-mortality rates for the Russian female population was nearly constant which—within the LC framework—would imply stagnation in the mortality dynamics for that period. This was not the case, however, as the stagnation in the average log-mortality was hiding the substantial and different dynamics at different ages: mortality at young ages was declining, while mortality at ages 35-70 was increasing at different speeds. These diverse dynamics were only reinforced in the late 1980s and the 1990s, when mortality decline at child ages and mortality increase at old ages were nearly continuing their previous trends, while mortality at young adult ages experienced dramatic perturbations and overall growth.

One may consider two additional types of correlation: correlations across time and correlations between sexes. A study of these correlations is not relevant to comparing the LC method with direct extrapolation. However, it might provide information important for developing both methods. Correlations of residuals from linear trends of age-specific log-mortality rates of male and of female populations are usually erratic. For shorter periods, these correlations are mostly insignificant for smaller countries (like Sweden and Austria), and for bigger countries they are significant (about 0.6-0.8) only if same-age cross-sex correlations are considered. Cross-time correlations are significant for bigger populations and same-age residuals close in time to each other. When the time lag between residuals becomes wider (more than about five years) or when a smaller population is concerned, these correlations are also usually insignificant. For longer data periods, they prove to be significant which, however, is a mere reflection of the existence of non-linear trends in longer time intervals. When correlations in rates of mortality decline are concerned, cross-time correlations are usually insignificant even in the case of long data periods.

Based on the analysis of correlations observed in data, one may draw the following conclusions:

*First*, the LC model does not seem to give a better fit to data compared to fitting separate trends. Correlations in residuals from separate trends are usually significant for adjacent ages only. It might be reasonable to incorporate these 'close-vicinity' correlations in projections. Nevertheless there are no reasons for pooling all age-specific dynamics into a single mortality trend. At the same time, persistent correlations within a broad age groups, if observed, may justify applying the LC method to subsets of the age span, not to the entire population[2]. As it seems, such correlations may be expected from historical data and also from data on countries where mortality is substantially affected by external causes of death, such as epidemics and famines.

*Second*, fitting separate trends allows flexibility in terms of using age-specific durations of data periods or of fitting different models at different ages. Fitting optimal trends to optimal data periods at different ages may significantly reduce extrapolation errors in the short run. The explicit study and projection of separate age-specific trends

[2] An example of improving the performance of the method in this way may be found in Lundström and Qvist (2004).

may also facilitate the usage of expert judgement (see examples above, in the literature review).

*Third*, cross-time correlations, which usually disappear with about a five years' lag, suggest that such correlations may be ignored in estimating trend parameters only if the data period covers not less than several decades. (So that observations from different parts of the data period will be less correlated, which may preclude from biases in parameter estimation.) It may also be necessary to use OLS procedures with autocorrelations explicitly taken into account.

*Fourth*, for the same reason as above, certain peculiarities of the most recent observation ('*jump-off*' year) might be important to take care of when projecting the nearer future (five to ten years); yet for a longer-range projections their effect may be neglected[3].

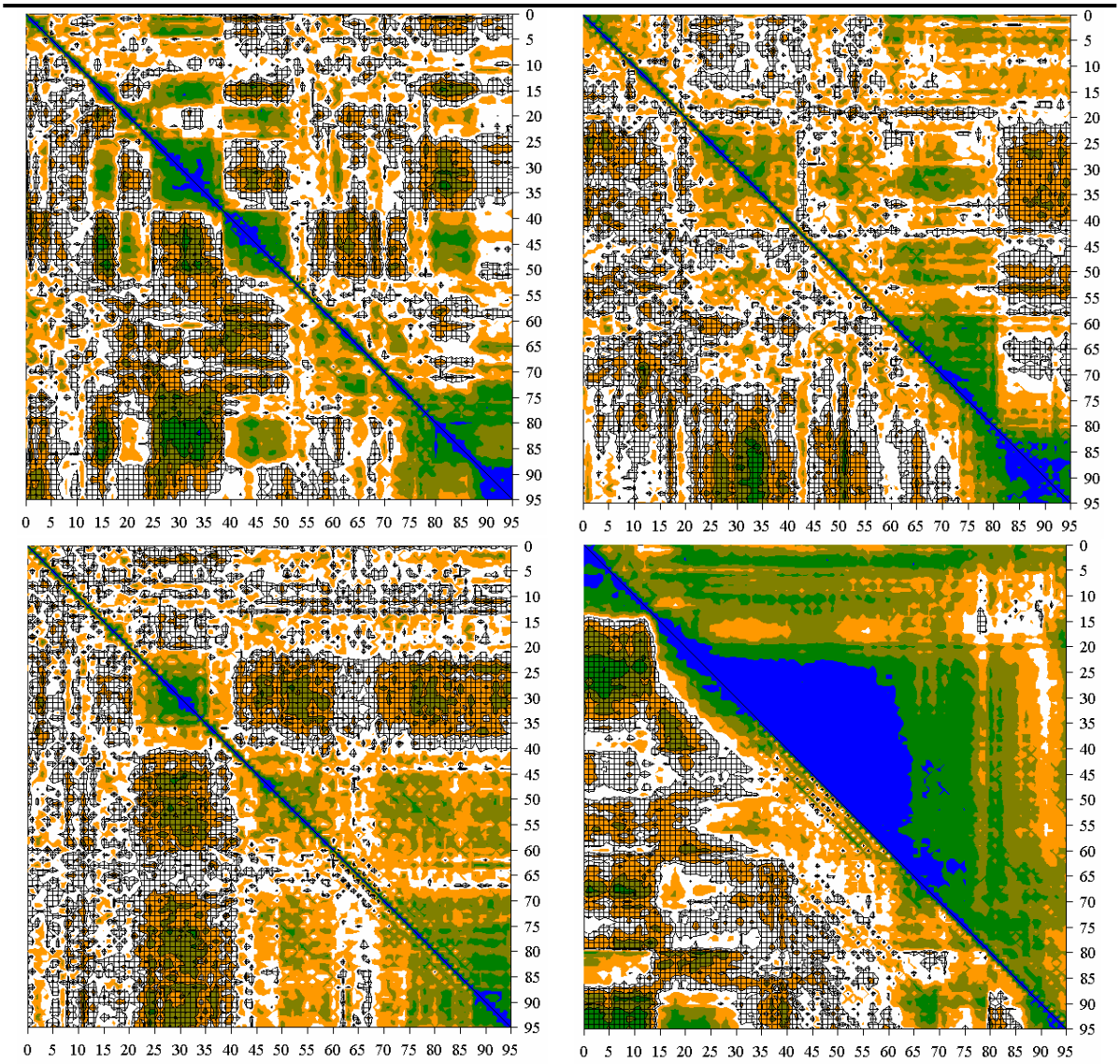
*Fifth*, cross-sex correlations may usually be neglected as well in estimating the trends' parameters.

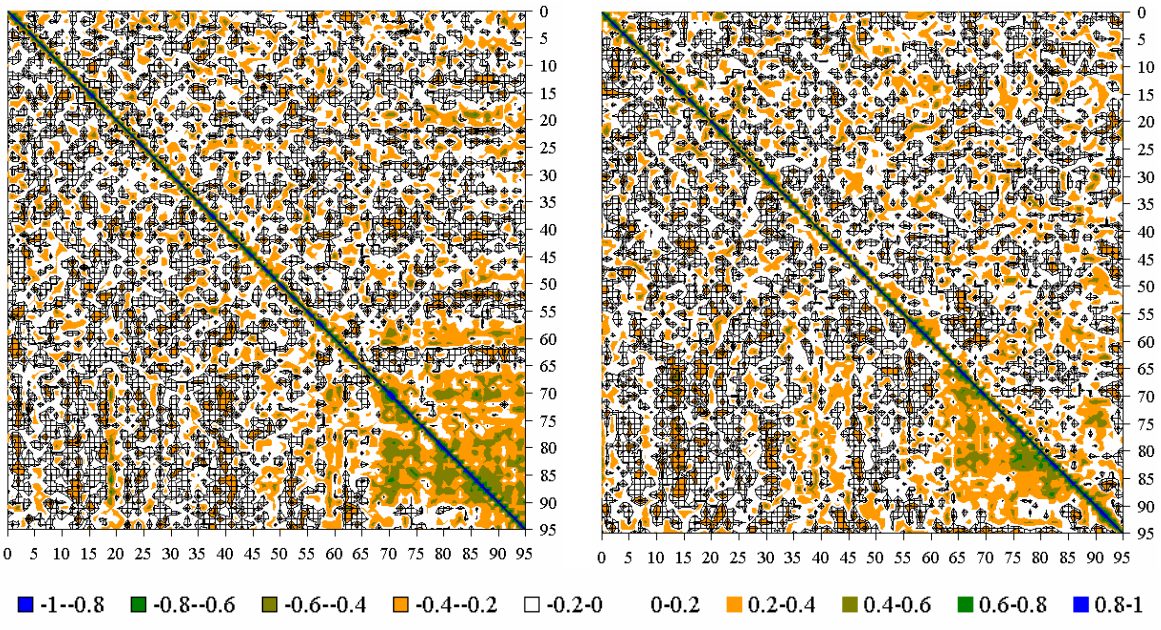
*Sixth*, age/sex-specific trends, being *estimated* separately, may not be *projected* independently. Measures have to be taken to reconcile consistency in projections, which otherwise might be lost due to the continuation of temporary divergent developments in data[4]. In a way, the reconciliation procedures may be considered as a method of *modelling* the 'loose correlations' observed in the data and mentioned above.

[3] Indeed, the cross-time correlations might still be important for a probabilistic projection, as they might affect the typology of possible future paths.
[4] Examples of adjustments aimed at enforcing consistency are provided above; see also Ediev (2007) and further down in the paper.

**Figure 1**

Cross-age correlations of residuals of the LC model (1) (lower-left triangles) and of the linear model (2) (upper-right triangles) fit to the period 1980 to 2004 (females: USA, Japan, France, Russia, Sweden, and Austria in consecutive order).

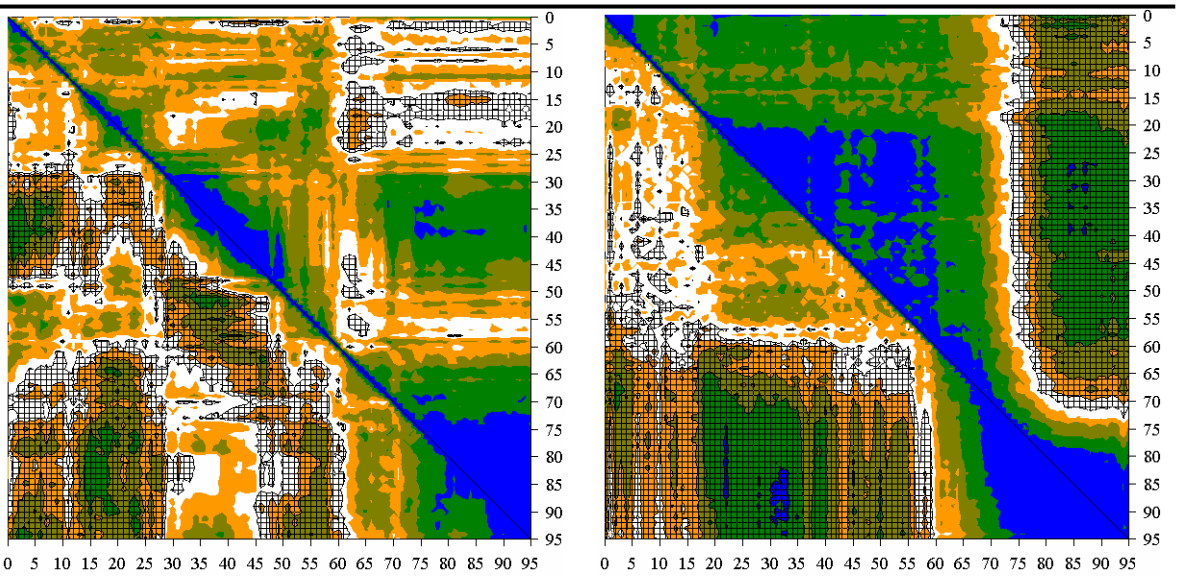


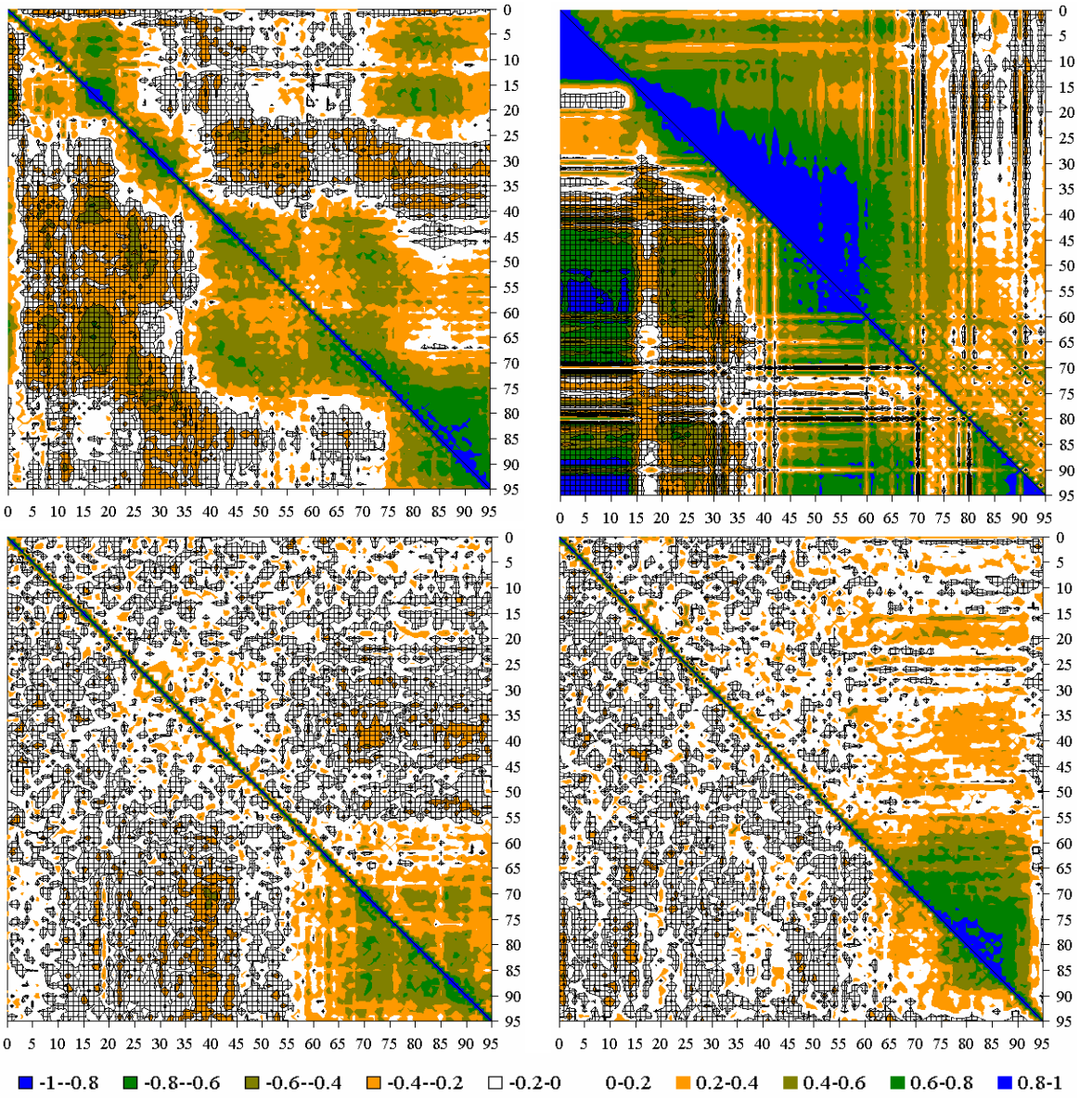



Source: Human Mortality Database, Ediev and Gisser (2007).

**Figure 2**

Cross-age correlations of residuals of the LC model (lower-left triangles) and of the linear model (2) (upper-right triangles) fit to the period 1960 to 2004 (females: USA, Japan, France, Russia, Sweden, and Austria in consecutive order).

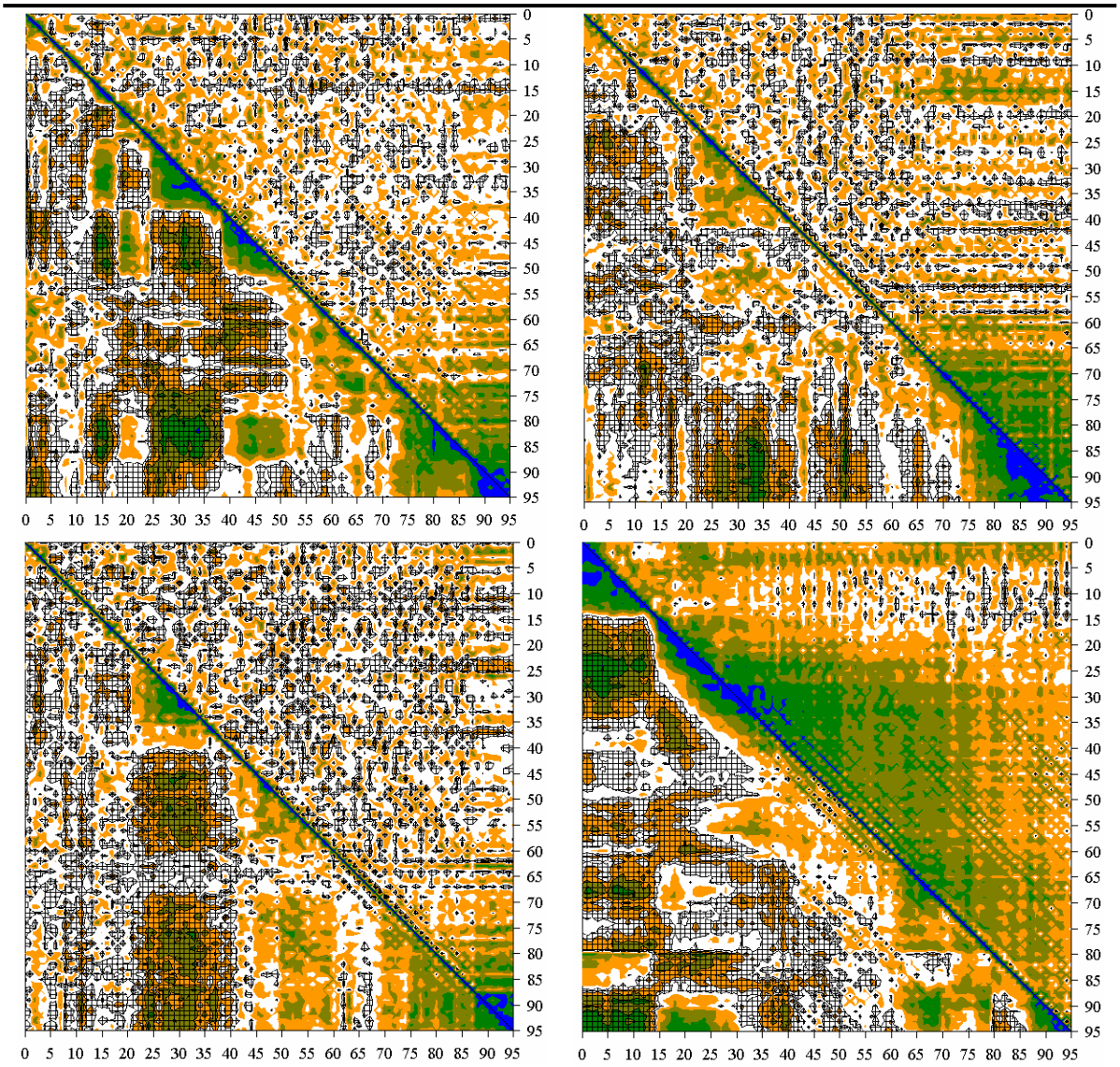


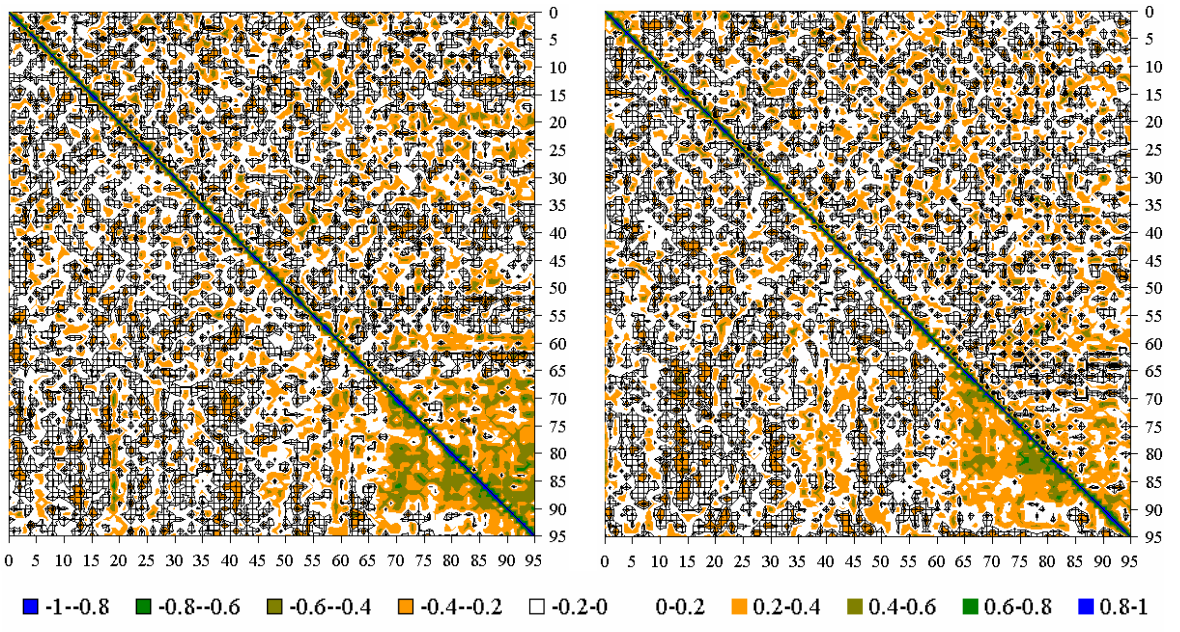



Source: Human Mortality Database, Ediev and Gisser (2007).

**Figure 3**

Cross-age correlations of residuals of the LC model (lower-left triangles) and of rates of mortality decline (upper-right triangles) in 1980 to 2004 (females: USA, Japan, France, Russia, Sweden, and Austria in consecutive order).

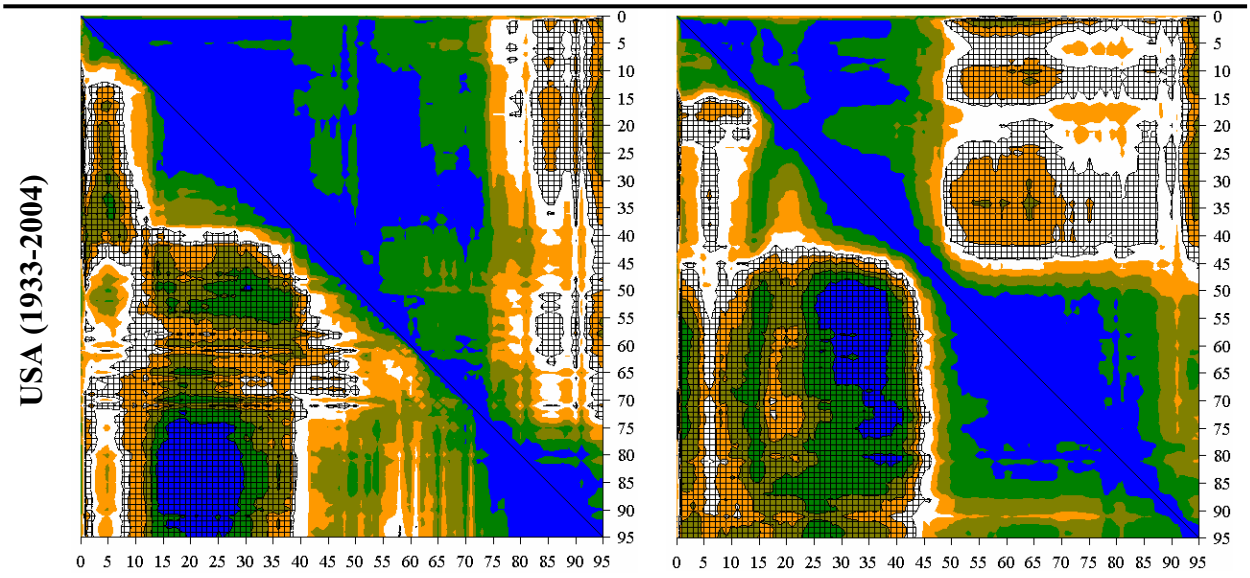


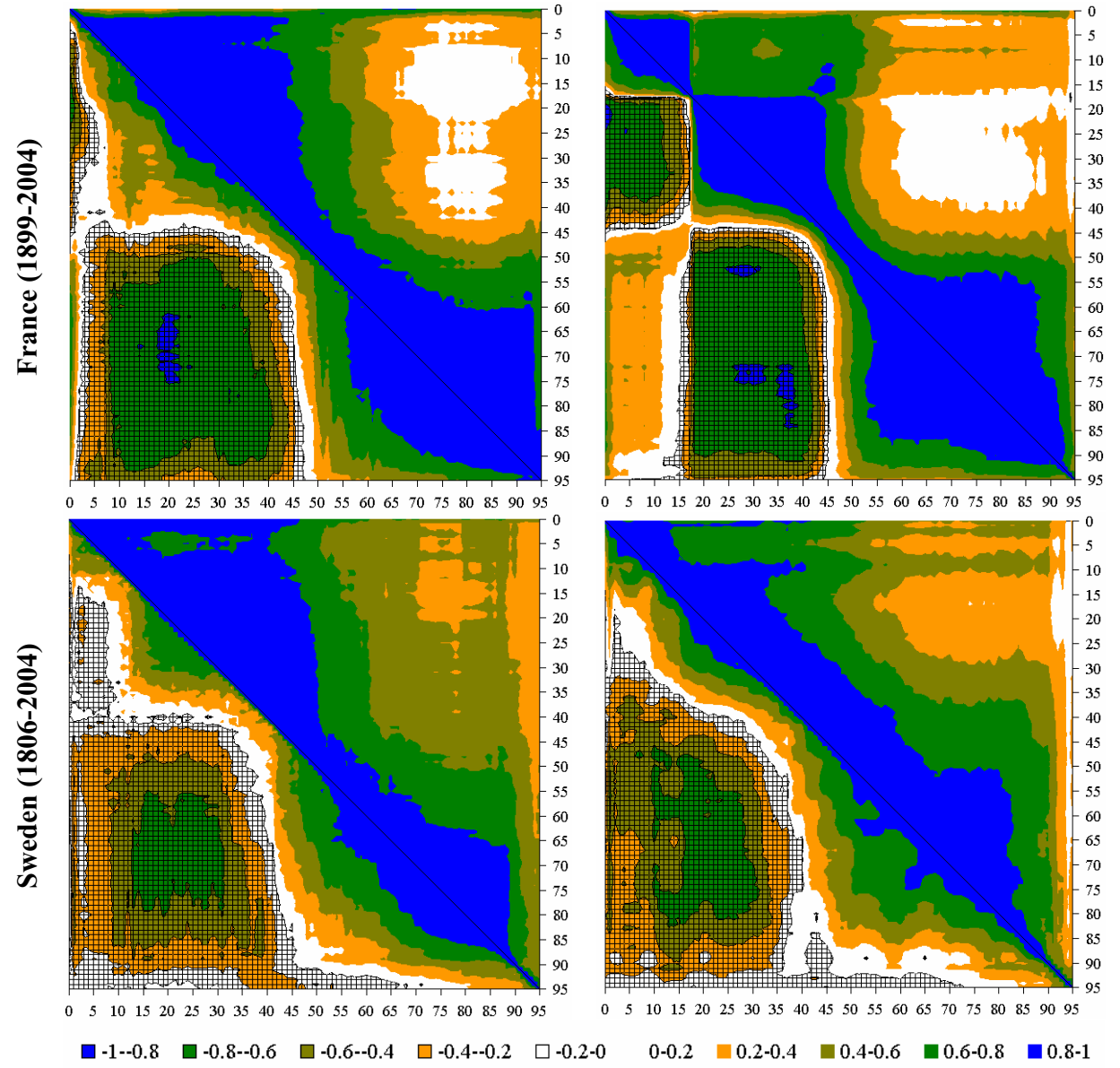



Source: Human Mortality Database, Ediev and Gisser (2007).

**Figure 4**

Cross-age correlations of residuals of the LC model (lower-left triangles) and of linear trends (upper-right triangles): females (left column) and males (right column).

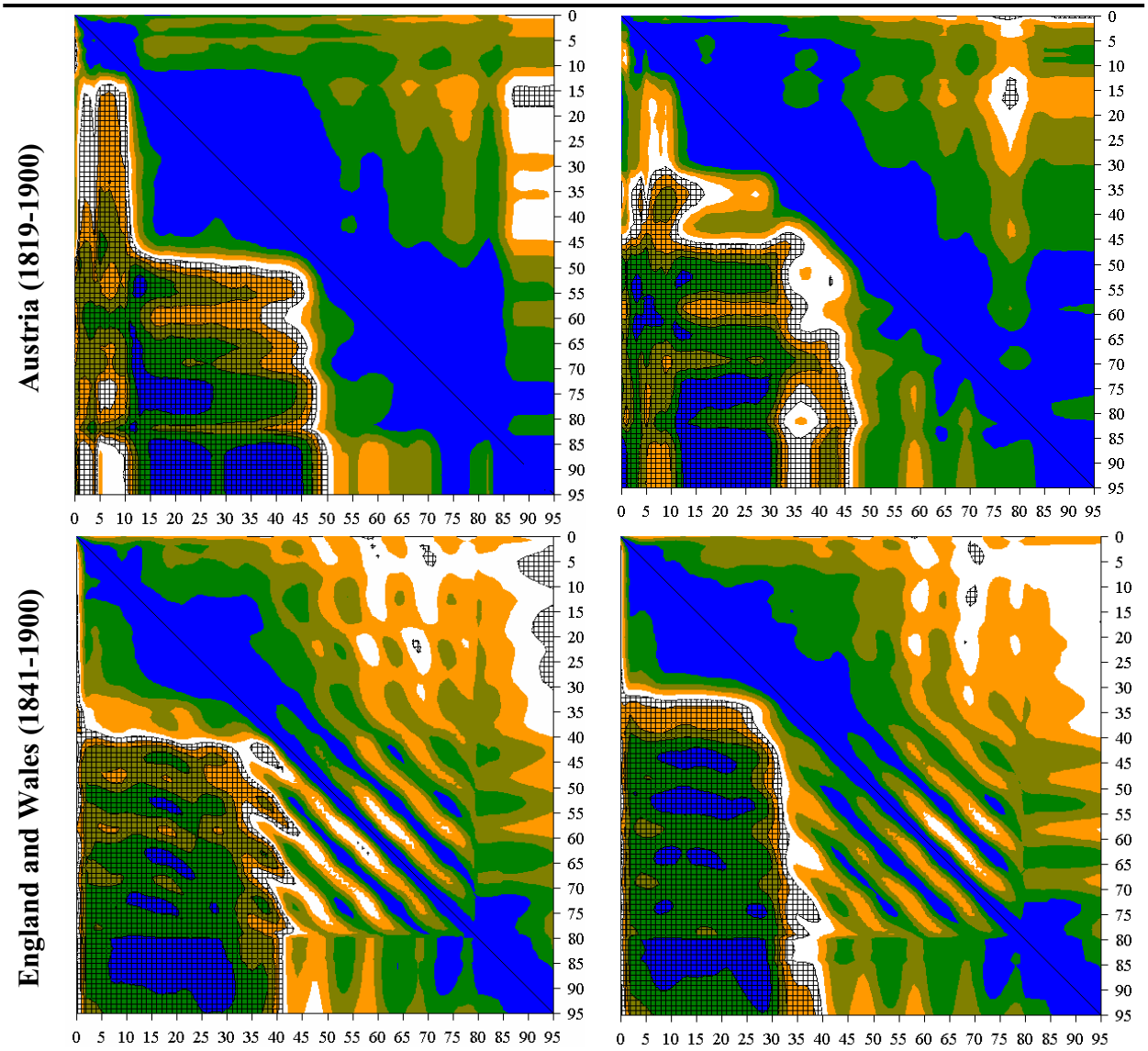


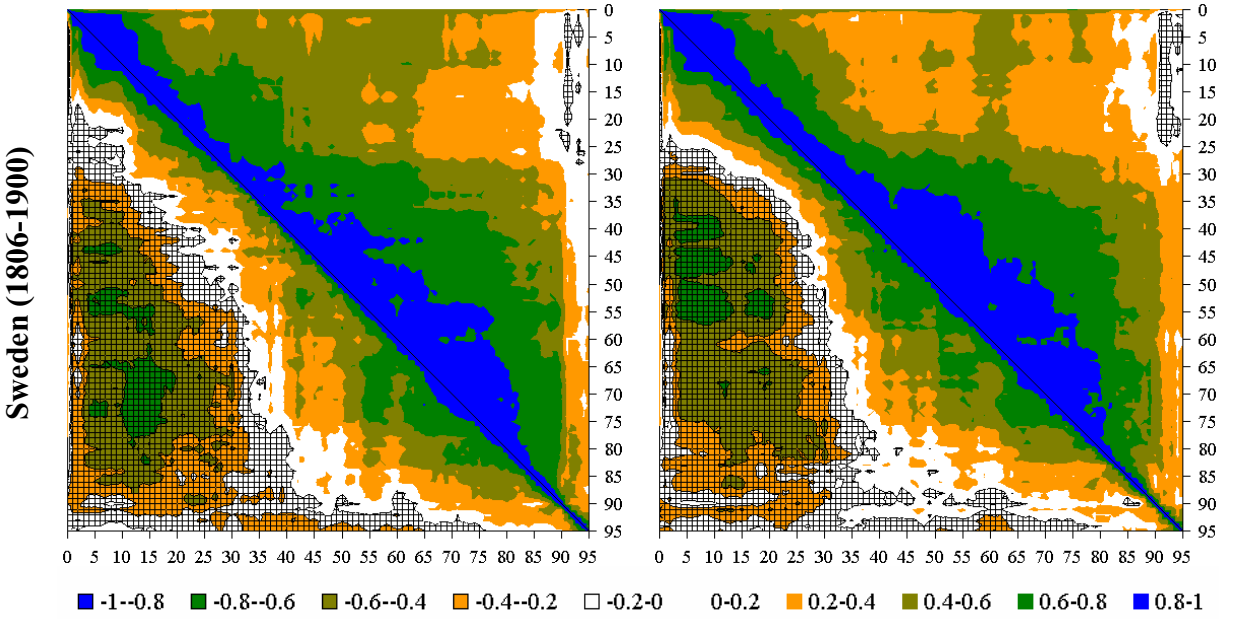



Source: Human Mortality Database, Ediev and Gisser (2007).

**Figure 5**

Cross-age correlations of residuals of the LC model (lower-left triangles) and of linear trends (upper-right triangles): females (left column) and males (right column).






Source: Human Mortality Database, Ediev and Gisser (2007).

### 2.2. Forecasting

Most of the similarities and dissimilarities mentioned above are also relevant to extrapolating. When the trends are assumed to be *linear*, both methods are essentially equivalent. In the case of non-linear trends, separate extrapolation might provide a better tool for projection. Hence, when projecting a *central* scenario as a continuation of past trends, procedures based on separate extrapolation are either equivalent to, or more suitable than, the LC method. When *deviations* from the central trend are assumed, however, the two methods may represent different approaches. These deviations may usually be of two types: due to altering projection scenarios in deterministic forecasting and also due to random variations in stochastic forecasting.

In the LC setup, these deviations will be produced in a collinear way. Despite being inconsistent with past data, such co-linearity in projections might be justifiable. In the case where projection scenarios are constructed using the method, *future* age-specific deviations from *past* trends may indeed be correlated as changes in trends can affect all age groups in similar ways (unprecedented overall mortality improvement or unexpected stagnation may take place). Similarly, when the method is used to produce probabilistic forecasts, it may be interpreted as modelling *forecasting errors* rather than the actual volatility of log-mortality rates in the future. And errors might be much more correlated than deviations from trends, as for instance the failure to predict overall mortality improvement or stagnation in the future could have a similar effect on prediction errors at all ages. At the same time it is clear that the LC method provides a stylized picture of the future, as it assumes that (i) age-specific trend changes and forecasting errors are completely correlated, without any allowance for additional uncorrelated dynamics; (ii) the trend

changes and forecasting errors are related by the same coefficients of proportionality, which are relating slopes of the trends of age-specific log-mortality rates in the past.

To some extent, validity of these two assumptions may be examined based on past data. For an example of how actual correlations in trend changes might look like, one may refer again to correlations between residuals of age-specific linear trends of log-mortality rates for a possibly long data period, during which there were indeed changes in trend slopes. An example of that kind is presented in Fig. 4 (left column). Indeed, linear trend changes were more correlated compared to residuals of adequate (possibly, non-linear) trends. Yet, the pattern is also not supportive for assumption of complete co-linearity.

The second assumption of the method, i.e., the assumption about the proportionality of future relative trend changes and of forecasting errors to trend slopes themselves in past, may be checked by comparing measures of trend slopes estimated from the data to measures of age-specific deviations from those trends (hence, considering past deviations from the overall linear trend to be proxies for magnitudes of trend changes or for errors of a best possible linear 'forecast' of past data). We carry out such an analysis by comparing linear trend slopes (both estimated from separate linear trend fitting and also from the LC model) to standard deviations of data from separate age-specific linear trends in past. In order to facilitate the comparison, we normalize all these indicators dividing by their average values for the entire age span. (The slopes' function $b_x$ of the LC model is already normalized in such a way that, by the scaling definition, the average value of $b_x$ is set to one.) Ideally for the LC model, these normalized functions should coincide. Empirical data (Figure 6) indicate that the LC model's assumption about equality between relative slopes and the relative uncertainty of age-specific dynamics of log-mortality rates *is valid* as a first approximation. Possibly this may add an explanation of the method's high efficiency in conducting probabilistic projections. At the same time, this assumption is only a rough representation of the overall tendency of log-mortality volatility to decrease with age. The assumption typically overestimates the uncertainty at childhood ages and at ages around 60 to 70 years. Also in most cases it underestimates the uncertainty at ages around 20 to 30 and, quite important for projecting the future prospects of mortality reduction, at the oldest ages (around 80 to 90 years and above). At these oldest old ages where mortality has been almost stagnating in the past, i.e., slopes of past trends were around zero, the LC method based on the aforementioned assumption assumes only a negligible uncertainty in the future, i.e., a continuation of the mortality stagnation at these ages. At the same time, the volatility of mortality rates at these ages was usually not lower but sometimes higher than at younger ages. Apart from possible problems with the quality of vital statistics at these ages (considering the usually small population numbers observed), this may well reflect unstable trends due to ongoing changes in old-age mortality in the contemporary period. In any case, there is no ground for assuming much less uncertainty in the future mortality at oldest old ages than, say, at ages around 70. An additional factor undermining the uncertainty at the level of individual age groups lies in the fact that the LC method builds forecasting uncertainty on stochastically modelling the dynamics of an aggregate variable, namely, the $k_t$ function. If age-specific log-mortality rates were to correlate completely, uncertainty in the overall mortality index would be relevant to the uncertainty at each individual age group. When individual rates' dynamics are less correlated, however, there

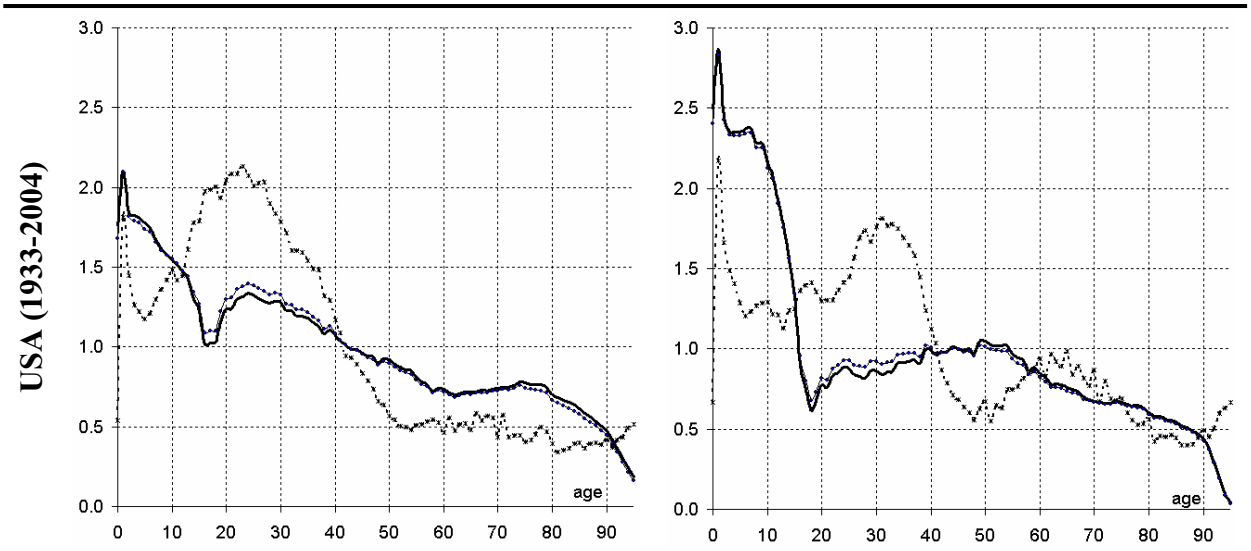
will be more uncertainty at individual age groups at any given level of uncertainty for an aggregate index of mortality.

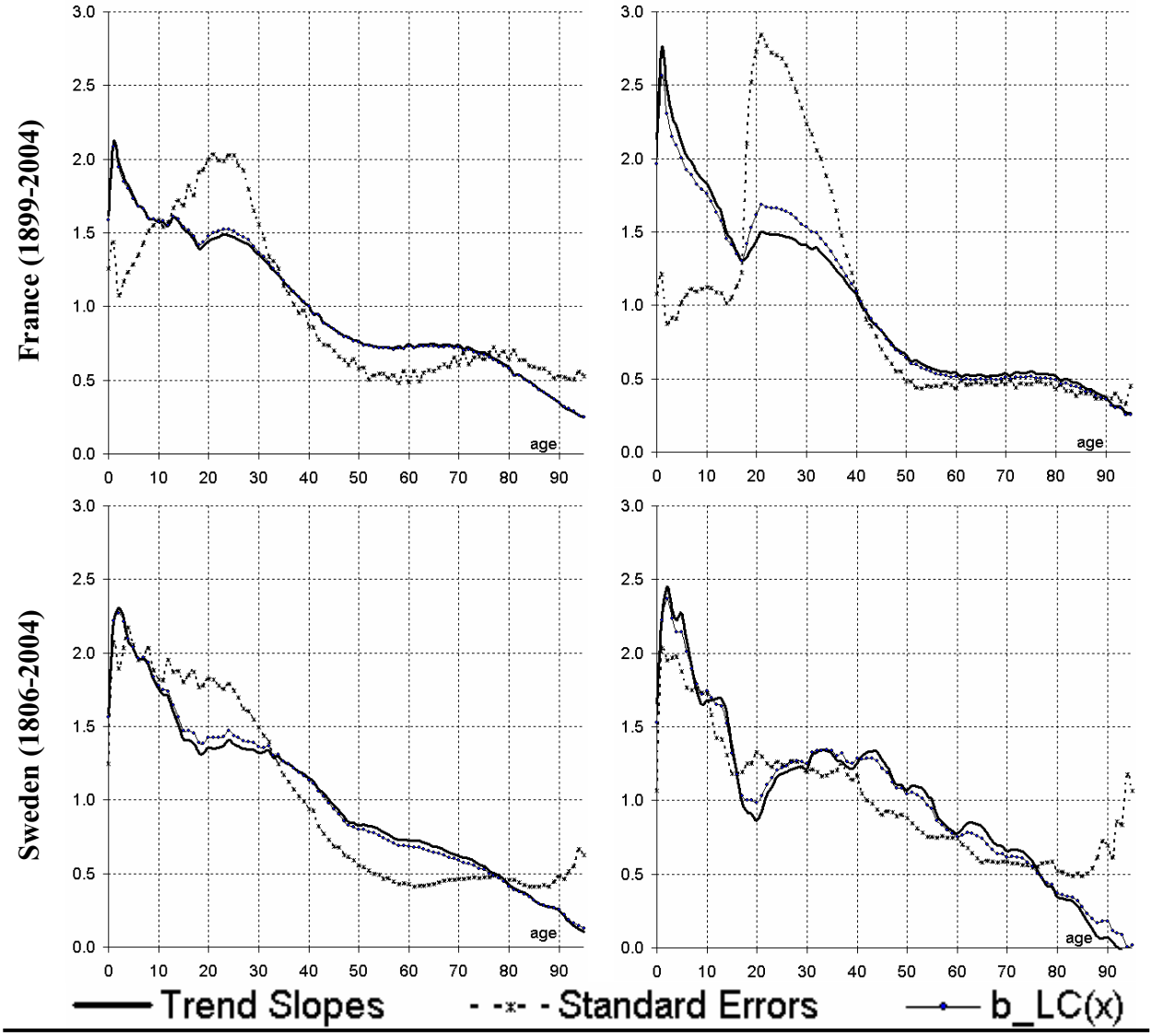
Another observation from Fig. 6 is related to the estimation issues discussed above: due to the lack of correlations among trend residuals, LC estimates are quite close to separate estimates of trend slopes, only slightly shifting towards relative standard errors at ages when these errors are considerably different from relative trend slopes.

With regard to the ability of separate extrapolation to address future uncertainty, in its simplest form this method is less effective than the LC method. Indeed, for a standard regression model there are traditional explicit estimates of standard errors in forecasts, which is an advantage of the method. Yet on the one hand these errors—estimated for each age/sex group separately—do not provide any insight into possible correlations between errors. And another shortcoming of the traditional methods to extrapolate trends is their inability to explicitly address the possibility of trends to *change* in the future. Meanwhile the data always show changes in linear trends and it should be expected to have such phenomena in the future as well. In the LC approach, this is indirectly reflected by using the random walk with drift procedure when generating probabilistic forecasts. This may be seen as another extreme, however, as the data do not show any repetitive annual breaks in their trend.

**Figure 6**

Age-specific linear trend slopes, standard errors of linear trends, and LC slope parameters $b_x$ normalized by dividing by the average value for the entire age span: females (left column) and males (right column).






Source: Human Mortality Database, Ediev and Gisser (2007).

## 3. Towards an Alternative: Designing the Central Scenario Projection

The comparative discussion of the two methods presented here may provide a basis for developing a new approach combining the advantageous features of both. First, it seems more reasonable to start with separate studies of age/sex-specific mortality trends. Second, however, it should be taken into account that these trends—being prolonged without any changes into the future—may lead to implausible age patterns of mortality. Therefore, special correction procedures may have to be developed in order to preserve the internal consistency of projected mortality profiles. Third, when introducing uncertainty into the projection this should also be done in a consistent way, linking deviations of projected age/sex-specific rates from their central scenarios to each other (the LC method proposes an example of that kind). In this paper, we consider only the first two stages of the method, which are concerned with building the central scenario of the projection. Aspects of

addressing uncertainty by developing high and low scenarios and also by conducting probabilistic projections will be addressed elsewhere.

To facilitate the reading, we start with a sketch overview of the method.

*General overview of the method*

3.1. Estimation of parameters of separate age/sex-specific linear trends.

*- Estimation of optimal (age/sex-specific) durations of most recent data periods consistent with the linearity of the trend.*

*- Estimation of cross-time autocorrelations of residual terms.*

*- Estimation of trend parameters ($a_x$, $b_x$) by weighted Least Squares method.*

*- Estimation of the standard errors of the estimates.*

3.2. Estimation of a long-term plausible schedule of mortality decline rates $b_x^*$.

*- Estimation of the monotonic $b_x^*$ function by the Min-Max method (Ediev 2007) or by a method of optimal-fit to $b_x$.*

3.3. Calculation of convergence parameters.

*- Parameter of convergence to trend from the last observation – based on cross-time autocorrelations.*

*- Parameter of convergence of $b_x$ to the long-term consistent schedule of mortality decline rates $b_x^*$ – based on estimated optimal durations of data periods with linear trends.*

3.4. Projection.

Following, these steps of the method are described in details.

### 3.1. Estimation of Parameters of Separate Age/Sex-Specific Linear Trends

As it is supported by the analysis of correlations, a separate estimation of age-specific trends seems to be a better approach. Although non-linear trends might be of value in some cases, we present here only the method based on fitting *linear* trends, i.e., on model (2).

Let us notate $\eta_{x,t} \overset{def}{=} \ln\left[m(x,t)\right]$. Model (2) may then be rewritten as follows:

$$\eta_{x,t} = a_x + b_x t + \varepsilon_{x,t}. \qquad (3)$$

Linear trends are not always relevant to the entire data set. Therefore only the recent period of data, which supports linearity assumption, is to be used in parameter estimation. (Among other things, this will reduce forecasting errors in the short run.) Testing some goodness-of-fit statistics might help in determining the necessary duration of the recent

data period. However, we use a simpler approach which works quite effectively in practice. Formally speaking, one should take into account possible autocorrelations of errors $\varepsilon_{x,t}$ to estimate parameters in (3). For a simplified method of detecting the most recent optimal data period with a linear trend, however, these correlations are ignored. (Yet, bearing them in mind it is imposed that the data period must not be shorter than 20 years.) For ordinary Least Squares (LS) estimates of parameters in (3), the standard forecasting error at time $t$ is given by the formula:

$$\sigma_{\eta(x,t)} = \sqrt{E\left(\eta_{x,t} - \hat{\eta}_{x,t}\right)^2} = \sqrt{E\left(\eta_{x,t} - \left(\hat{a}_x + \hat{b}_x t\right)\right)^2} = \sigma_\varepsilon \sqrt{1 + \frac{1}{n} + \frac{\left(t - \bar{t}\right)^2}{n\left(\overline{t^2} - \bar{t}^2\right)}}, \tag{4}$$

where hereinafter, cups denote estimates and forecasts; upper lines denote arithmetic averaging over the data period; *n* is the number of observations used in estimation; and $\sigma_\varepsilon$ is the standard deviation of the error term in (3), which may be estimated from residuals after fitting the linear model. For estimating the longest possible recent data period with a linear trend, the following relative absolute deviations from the trend are analyzed for each possible beginning year $t$ of the period:

$$\varphi_{x,t-d} = \frac{\left|\eta_{x,t-d} - \hat{\eta}_{x,t-d}\right|}{\hat{\sigma}_{\eta(x,t-d)}}, \tag{5}$$

where *d* is the lag used in checking whether the observation at year $t-d$ fits to the trend observed after $t$. At $d=1$, the procedure would provide a check of fitness to the trend of the observation right before the first observation included in the data period. This choice, however, is not feasible as for a short time lag one cannot distinguish between trend change and random deviation from the trend. Although the optimal choice for *d* could be a function of the expected magnitude of trend change and of errors (4), we simplify the procedure by setting $d=10$. (However, $d = \min\left(10, t - t_{\min}\right)$ when $t$ approaches the very first year of observation to avoid extending beyond the available data.) The following simple rule is used to detect the optimal starting year for the data period:

$$t_{start,x} = \min_{t \le t_{\max}} \left\{ t : \varphi_{x,t-d} \ge \varphi_{\max} \right\} - 1, \tag{6}$$

where $\varphi_{\max} = 2$ is the threshold value, above which the deviation is considered to be significantly inconsistent with the trend, and $t_{\max}$ is the latest possible year for starting the data period (we use the year 20 years prior the last observation, as mentioned above). Estimates (6) are additionally smoothed by applying 5-year moving averages to eliminate erratic variations. Some examples of such estimates are provided further down, in the bottom parts of the graphs presented in Figure 7.

Having detected the optimal data periods, the parameters for separate age/sex-specific trends may be estimated. The existence of autocorrelations—which is the case for countries with large population—may preclude from basing the estimation on short periods of data or from doing so without including these autocorrelations into the estimation procedures. One should also be aware that positive cross-time autocorrelations may significantly distort all autocorrelation estimates, even when such estimates are based on relatively long periods of data (when these autocorrelations at adjacent years are high enough, even a century-long data period might be insufficient to robustly estimate autocorrelations at longer time lags or between different age groups). For mortality data,

the duration of a period when the data exhibit a more or less suitable linear trend is several decades, which is too short for the distorting effects of autocorrelations to be neglected. Therefore it may not be considered reliable to estimate too much information on correlations from such data. For the reasons outlined, we propose here to obtain only a stylized correlation pattern for cross-time correlations and to neglect completely cross-age/sex correlations in the data. In particular, cross-time correlations are approximated by the following simplified model:

$$Correl\left(\varepsilon_{x,t},\varepsilon_{x,t-k}\right)=\rho^{k}, \tag{7}$$

where $\rho$ is the correlation coefficient for same-age/sex errors in adjacent years. This coefficient may be estimated based on residuals of linear trends preliminarily fit to a period when data presumably show linearity. (Note that the correlation parameter $\rho$ is an average over all age groups; therefore, the data necessary for its calculation will be abundant even for a relatively short period; and a period of one to three decades may usually fit the task.) An advantage of having the same estimate for all age groups is that this estimate may be considered to be virtually uncorrelated to data and estimations of the model parameters for each individual age group.

With autocorrelations taken into account, parameters in (3) may be estimated using the following matrix equation of the least squares method with autocorrelations:

$$\hat{\mathbf{a}}=\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix}=\left(X^{T}C^{-1}X\right)^{-1}X^{T}C^{-1}Y, \tag{8}$$

where the index of age group is omitted, index 'T' denotes transposition, and the following matrices are introduced:

$$X=\begin{pmatrix}1 & t_1\\ 1 & t_2\\ \vdots & \vdots\\ 1 & t_n\end{pmatrix},\ Y=\begin{pmatrix}\eta_{t1}\\ \eta_{t2}\\ \vdots\\ \eta_{tn}\end{pmatrix},\ C=\begin{pmatrix}1 & \rho & \rho^2 & \cdots & \rho^{n-1}\\ \rho & 1 & \rho & & \rho^{n-2}\\ \rho^2 & \rho & 1 & & \rho^{n-3}\\ \vdots & & & \ddots & \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & & 1\end{pmatrix}.$$

The correlation matrix has a reciprocal matrix with a tri-diagonal structure:

$$C^{-1}=\begin{pmatrix}1+\frac{\rho^2}{1-\rho^2} & -\frac{\rho}{1-\rho^2} & 0 & \cdots & 0 & 0\\ -\frac{\rho}{1-\rho^2} & 1+\frac{2\rho^2}{1-\rho^2} & -\frac{\rho}{1-\rho^2} & & 0 & 0\\ 0 & -\frac{\rho}{1-\rho^2} & 1+\frac{2\rho^2}{1-\rho^2} & & 0 & 0\\ \vdots & & & \ddots & & \vdots\\ 0 & 0 & 0 & & 1+\frac{2\rho^2}{1-\rho^2} & -\frac{\rho}{1-\rho^2}\\ 0 & 0 & 0 & \cdots & -\frac{\rho}{1-\rho^2} & 1+\frac{\rho^2}{1-\rho^2}\end{pmatrix}. \tag{9}$$

Combining (8) and (9), it can be shown that

$$\hat{b} = \frac{\overline{yt} - \bar{y}\cdot\bar{t} + c\dfrac{n-1}{n}\overline{\Delta_y}}{\overline{t^2} - \bar{t}^2 + c\dfrac{n-1}{n}}, \qquad \hat{a} = \bar{y} - \hat{b}\bar{t}, \tag{10}$$

where $c = \dfrac{\rho}{1-\rho^2}$ and $\Delta_{yt} = y_{t+1} - y_t$. Relations (8), (10) turn into conventional OLS formulas when autocorrelations are neglected, i.e., $c = 0$.

Estimates (8), (10) are unbiased, $E\hat{\mathbf{a}} = \mathbf{a}$, and covariances of parameter errors are given by

$$E\left[(\hat{\mathbf{a}} - \mathbf{a})(\hat{\mathbf{a}} - \mathbf{a})^T\right] = \sigma^2\left(X^T C^{-1} X\right)^{-1} = \frac{1}{n}\frac{\sigma^2}{\overline{t^2} - \bar{t}^2 + c\dfrac{n-1}{n}}\begin{pmatrix} \overline{t^2} + c\dfrac{n-1}{n} & -\bar{t} \\ -\bar{t} & 1 \end{pmatrix}, \tag{11}$$

where $\sigma^2$ is the dispersion of the error term in (3), unbiased estimate for which is given by the following expression (see self-sufficient derivation in the Appendix):

$$\hat{\sigma}_\varepsilon^{\,2} = \frac{\hat{E}^T C^{-1}\hat{E}}{n-2} = \frac{1}{(n-2)(1-\rho^2)}\left(\sum_{i=1}^{n} e_i^{\,2} + \rho^2\sum_{i=2}^{n-1} e_i^{\,2} - 2\rho\sum_{i=1}^{n-1} e_i e_{i+1}\right), \tag{12}$$

where $e_i = y_i - \hat{y}_i$ - are the residuals of the model. From (11), (12), unbiased estimates for quadratic errors of the parameters' estimates are given by the following expressions:

$$\hat{\sigma}_a^{\,2} = \frac{1}{n}\hat{\sigma}_\varepsilon^{\,2}\left(1 + \frac{\bar{t}^2}{\overline{t^2} - \bar{t}^2 + \dfrac{\rho}{1-\rho^2}\dfrac{n-1}{n}}\right), \tag{13}$$

$$\hat{\sigma}_b^{\,2} = \frac{1}{n}\hat{\sigma}_\varepsilon^{\,2}\frac{1}{\overline{t^2} - \bar{t}^2 + \dfrac{\rho}{1-\rho^2}\dfrac{n-1}{n}}. \tag{14}$$

### 3.2. Estimation of the Long-Run Plausible Schedule for Mortality Decline Rates $b_x^*$

As described above, a drawback of extrapolative methods for projecting age/sex-specific mortality rates is that age/sex-specific trends, being extrapolated independently, may produce quite implausible age/sex patterns of mortality. Mainly, this implausibility arises from the slopes $b_x$, which, being kept at separately estimated values, may lead to implausible divergent mortality dynamics in the projection. Therefore we develop special procedures for correcting the estimates of the slopes in order to guarantee the consistency of the projection in the long run. Apart from conferring consistency to the projection, the corrections must improve the forecasting efficiency in general, as they utilize additional empirical knowledge about regularities in age structure of mortality which is neglected when estimating age-specific mortality trends separately. At the same time, the corrections

must be carried out so as not to reduce the forecasting efficiency at individual ages. In particular, there should be no significant correction when projecting to the nearest future, while corrections should be more pronounced in the long run when changes of trends for individual ages are more likely.

The idea is to assume that age-specific slopes $b_x$ gradually converge to eventual slopes $b_x^*$, the latter being constructed so as to grant the plausibility of the long-term mortality dynamics. Based on the general tendency of slopes $b_x$ to increase with age, which can be observed from the data, we also assume $b_x^*$ to increase monotonically with age (and also to be higher for males compared to females at a similar age). A simplified Min-Max method for deriving such a monotonic function was proposed earlier (Ediev 2007). Here we use another method[5] based on obtaining consistent slopes $\hat{b}_x^*$ as a solution for the following optimization problem, which can be solved, e.g., using the *dynamic programming* method[6]:

$$Z\left[b_x^*\right] = \sum_{x=0}^{X} \frac{\left(\hat{b}_x - b_x^*\right)^2}{\hat{\sigma}_{bx}^2 + \breve{\sigma}_*^2} \to MIN \tag{15}$$

under constraints:

$$\begin{cases} \hat{b}_{x+1}^* \geq \hat{b}_x^*, \\ \hat{b}_{x,MALES}^* \geq \hat{b}_{x,FEMALES}^*, \end{cases} \tag{16}$$

here $\breve{\sigma}_*^2$ is a preliminary estimate of the dispersion of residuals $\delta_x = b_x - b_x^*$ (as a rough estimate, we use a standard error of the linear trend fit to $\hat{b}_x$). One may also use additional constraints: $b^{Min} \leq \hat{b}_x^* \leq b^{Max}$, with $b^{Min}$ and $b^{Max}$ selected from additional considerations (e.g., it might be plausible to set $b^{Max} = 0$ in many cases to prevent a long-run mortality increase). Values of $\hat{b}_x^*$ obtained from (15) usually form a piece-wise constant function and may be smoothed by a moving averages procedure:

$$\tilde{b}_x^* = \frac{1}{2m+1} \sum_{y=x-m}^{x+m} \hat{b}_y^*, \tag{17}$$

where $2m+1$ is the length of the smoothing frame set at 11 years in the paper (at boundary age groups this frame is shortened accordingly). Typical examples are presented in Figure 7, where slope estimates $\hat{b}_x$ as well as the consistent slopes $\tilde{b}_x^*$ are presented for eight populations. The estimates are based on the most recent data periods with linear trend (see description above). The exceptional case of Russia is presented for comparison only;

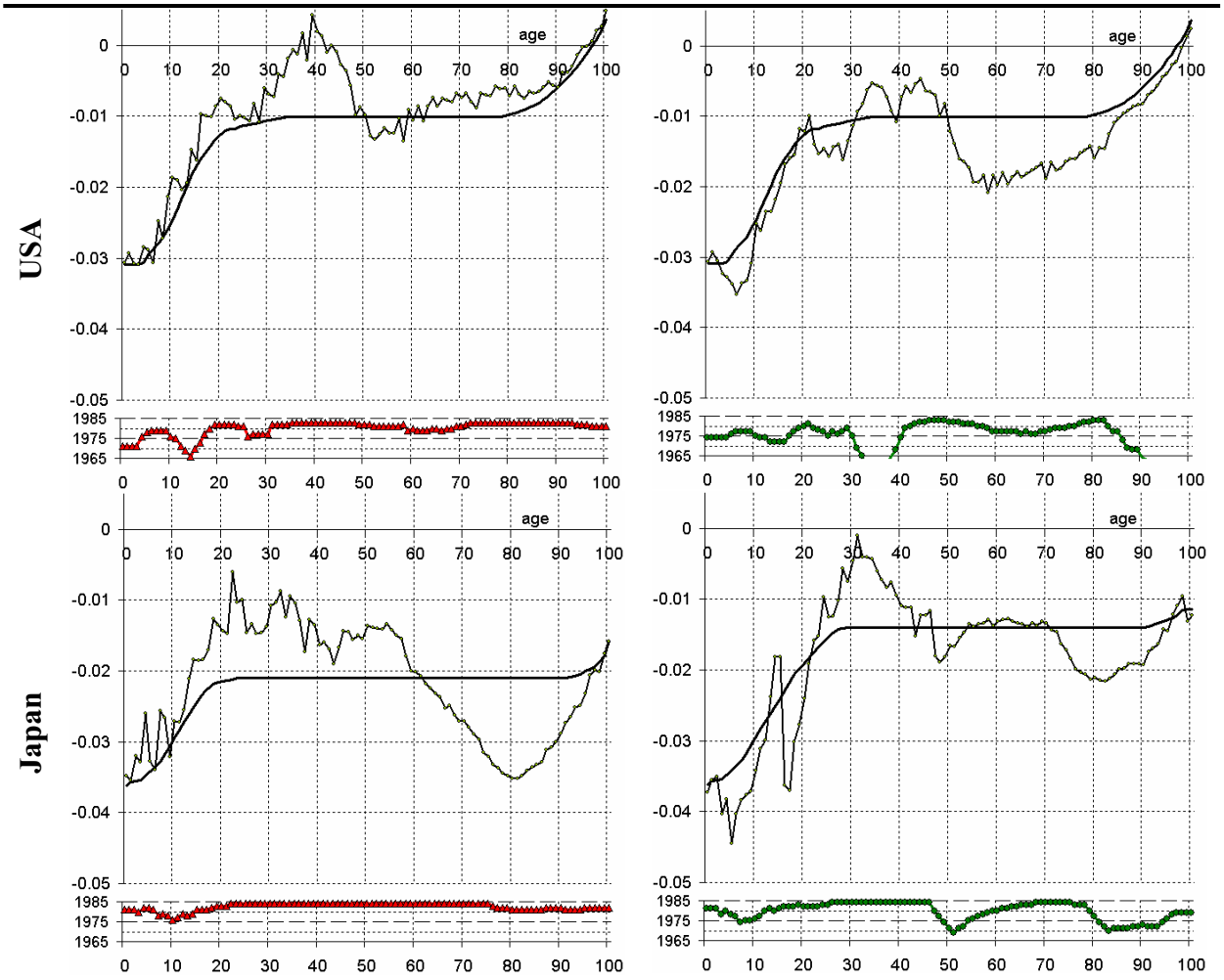
[5] Extensive simulations that were conducted suggest, however, that both the method used here and the previously proposed simpler Min-Max methods are in fact of similar performance.

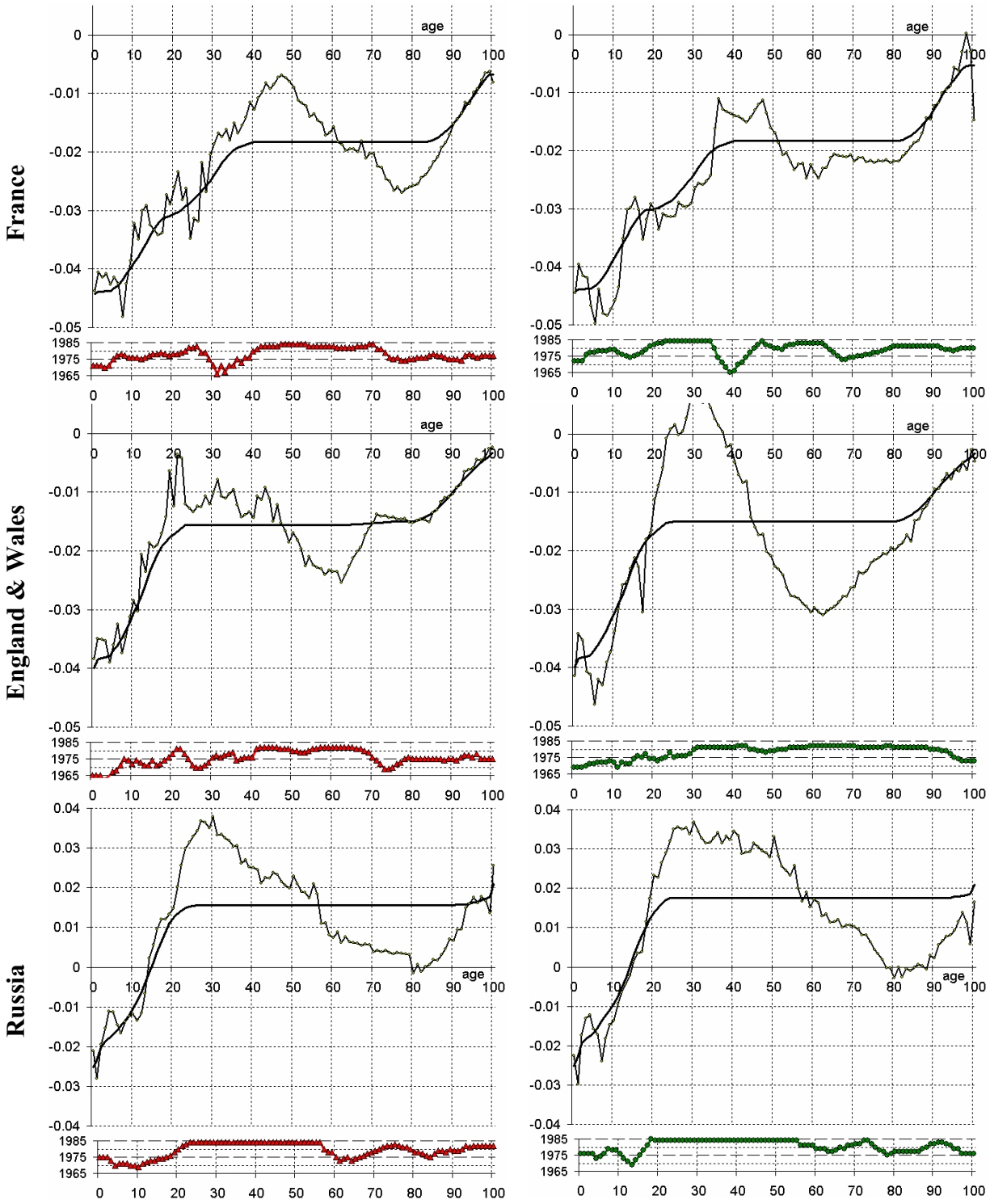
[6] Note that this procedure may probably be improved by introducing additional age/sex-specific weighting factors in order to reflect the relative importance of mortality dynamics at different ages for the dynamics of some general indicators of mortality, e.g., of life expectancy at birth. This would be similar to corrections applied in the LC method to the $k_t$ function in order to increase agreement of the model with the observed dynamics of life expectancy at birth or with the total number of deaths or—similarly—to using weighted least squares to estimate its parameters (Wilmoth 1993).

indeed, this case, where future mortality trends are not yet visible from the data, requires special treatment.

**Figure 7**

Slopes of age-specific most recent optimal linear trends and corresponding consistent slopes obtained from (15): females (left column) and males (right column). The bottom of each graph shows the age-specific optimal beginning years of the trend fitting period.




France
England & Wales
Russia
age
0
-0.01
-0.02
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-0.04
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0.03
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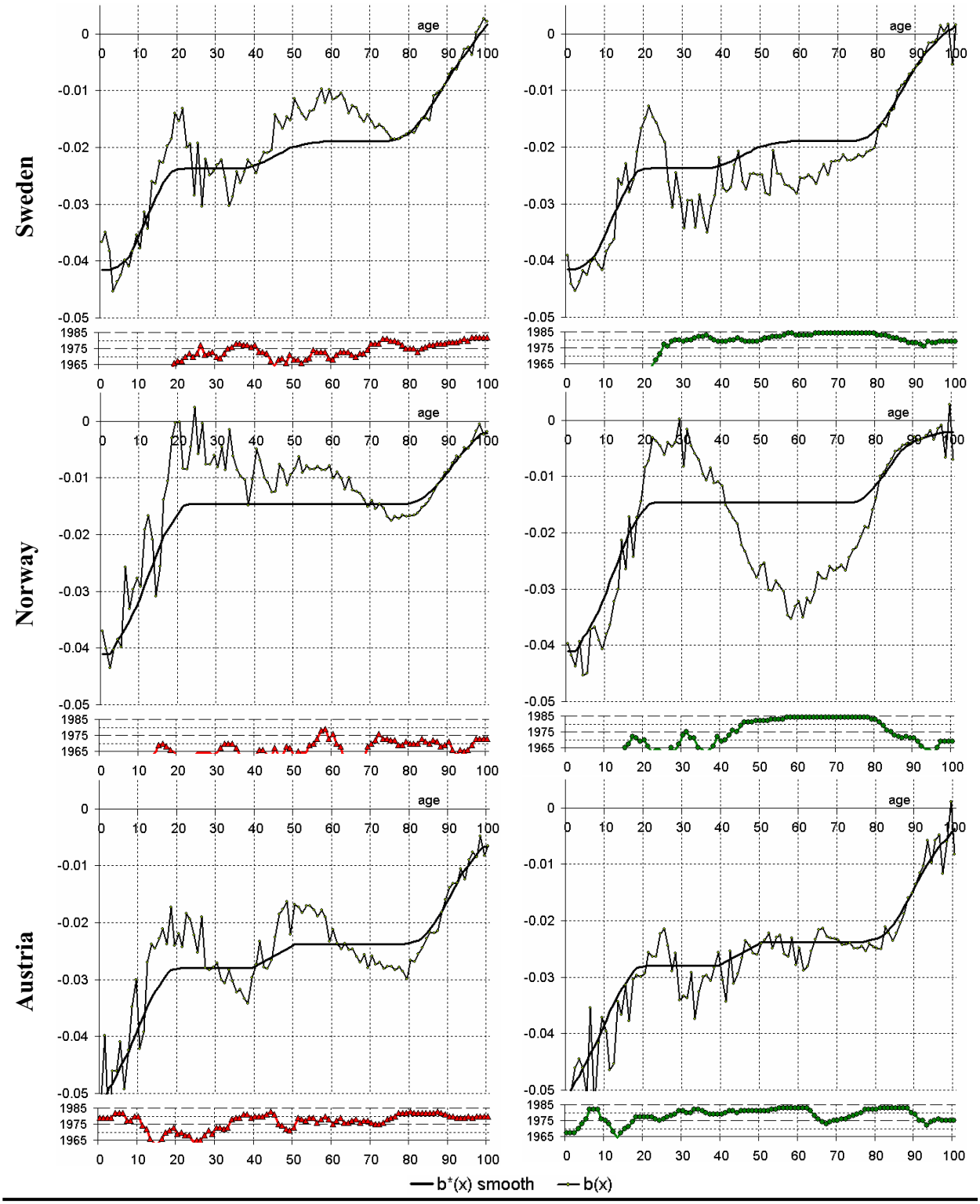
-0.03
-0.04
1985
1975
1965
0 10 20 30 40 50 60 70 80 90 100




Source: Human Mortality Database, Ediev and Gisser (2007).

### 3.3. Calculation of Convergence Parameters

The main idea of the method is to extrapolate the log-mortality rates as converging to their trends, which in turn converge to long-run trends determined by the consistent slopes $\hat{b}_x^*$. Consequently, we consider two types of convergence in the projection.

First, we model convergence of each age/sex-specific log-mortality rate from its most recently observed value ('jump-off value') to the trend. The speed of this convergence must depend on cross-time autocorrelations of residuals from linear trends in the past. Insignificant cross-time autocorrelation may suggest ignoring the jump-off value and immediately starting the projection from the trend value. On the other hand, high cross-time autocorrelations may suggest a gradual movement from the jump-off value to the trend. Under the assumed simple structure of these autocorrelations (7), one may use the following exponential model in a deterministic setup, which corresponds to AR(1) process for residuals in a stochastic setup[7]:

$$\varepsilon_{xt} = \rho \varepsilon_{xt-1}, \tag{18}$$

The second kind of convergence—of the trend slopes $\hat{b}_x$ to their long-run values $\hat{b}_x^*$—is modeled based on the observed durations of the most recent data periods consistent with linearity assumption, which are determined by (6). For all age/sex groups we assume that the expected duration of the period after which the trend is changing is

$$\lambda = \overline{t_1 - t_{start,x}}, \tag{19}$$

hereinafter by $t_1$ we denote the last year of the data period, i.e., the year preceding the projection period; note that we obtain (19) by averaging the observed durations over all age and sex groups. Further, we assume that the probability of a trend change at any given year is given by

$$\pi = 1/\lambda. \tag{20}$$

Correspondingly, in the deterministic setup we assume that deviations of the age/sex-specific slopes $\hat{b}_x$ from their eventual values $\hat{b}_x^*$ exponentially converge to zero[8]:

$$\hat{b}_{xt} - \hat{b}_x^* = (1-\pi)^{t-t1}\left(\hat{b}_{xt1} - \hat{b}_x^*\right). \tag{21}$$

### 3.4. Projection

Based on the model assumptions and parameter estimates discussed above, the central deterministic projection may be described as follows ($t = t_1 + 1, t_1 + 2, \ldots$):

[7] A possible improvement of the model could lie in introducing the cohort effect in (18), which we are neglecting here.

[8] The data indicate that the dynamics of deviations (21) may exhibit some cohort effects which this paper does not take into consideration, however.

$$
\begin{aligned}
\eta_{xt} &= \hat{\eta}_{xt} + \varepsilon_{xt}\,, \\
\hat{\eta}_{xt} &= \hat{\eta}_{xt-1} + \hat{b}_{xt-1}\,, \\
\varepsilon_{xt} &= \rho\varepsilon_{xt-1}\,, \\
\hat{b}_{xt} &= \hat{b}_x^{\,*} + (1-\pi)\left(\hat{b}_{xt-1} - \hat{b}_x^{\,*}\right).
\end{aligned} \tag{22}
$$

We illustrate the performance of the method by presenting projection results obtained both using the method itself and also by the LC method. First we present results for would-be projections from 1900 until the contemporary period based on data from the 19th century for three countries: Austria (Ediev and Gisser 2007), England and Wales, and Sweden (Human Mortality Database), see Figures 8-10. All three figures are constructed in a similar way. They contain four charts for projected and actual age-specific central death rates (per 1000, logarithmic scale) – two for females and two for males; and one chart representing the actual and the projected dynamics of life expectancy at birth for males as well as for females. Each chart containing graphs of death rates consists of four curves: death rates estimated from the original data for the base year (1900) and for the most recent year with available data (2005 for Austria and Sweden; 2003 for England and Wales) as well as curves of death rates projected to the last year by the direct extrapolation (DE) method (22) and the LC method. (Note that for the projection purposes we used the LC method based on the SVD procedure.) Charts in the first row represent results obtained by using the entire data set available for 19th century, i.e., we do not use estimates (6) in this case. Being based on similar long-period data, both methods provide similar results, which is natural in view of the aforementioned discussion of the methods, although a more detailed analysis reveals that the DE method produced lower infant mortality than the LC method. In both cases, both methods failed to predict the reductions in mortality which continued through the 20th century, although in the cases of England and Wales and of Sweden substantial reductions in mortality at young ages were indeed projected. The overall accuracy of both methods improves when parameters are estimated based on the optimal starting years of the data period (6), as indicated by the charts presented in the second row of each graph. (Since the LC method must be based on data with the same length at each age/sex group, we used the arithmetic average of age/sex-specific start years (6) as the start year for this method.) Despite the overall similarity in projections produced by both methods, two distinctions are notable. First, the LC method seems to be more likely to produce implausible profiles of mortality—concerning both the age structure of death rates and the sex differences in mortality. Second, even being based on a data period of optimal length, the LC method produced a much more moderately declining child mortality, which is in fact due to the monotonicity assumption (16) that was used in the DE method but not applied to the LC method. As a result, the LC method produced dynamics of life expectancy at birth with slower growth and—in the cases of Austria and Sweden—implausible crossovers, see the last charts in the figures.

The next projection exercise is focused on projections for the 21st century produced by both methods based on data from 20th century. For this exercise we have more countries with available data from the Human Mortality Database. Figures 11 to 18 contain graphs of projected age/sex-specific death rates and of the dynamics of life expectancy at birth and at age 65 for Austria, England and Wales, Sweden, the US, France, Japan, Italy, and Norway. (All calculations are based on the most recent optimal data periods, i.e., (6)

applies to the DE method and arithmetic average of estimates (6) – to the LC method.) Since the mortality decline in the late 20th century was even more dynamic than in the late 19th, the problems associated with implausibly forecasted mortality dynamics in future are stronger. The LC method, which is applied here without any consistency adjustments, often produces implausible mortality profiles in the long run. The cases of England and Wales, the United States, Japan and Italy are especially remarkable in this respect. One may also note that the both methods produce extremely low mortality rates at young ages by the end of the 21st century (0.001 per 1000 and less). On the one hand, such low levels may be considered unrealistic and some age/sex-specific low limits to mortality may be applied if there is enough biological or other evidence in support of such limits. On the other hand, such limits may also reflect our subjective biases due to the particular experience of observing mortality in 20th century only. Examples of the kind presented in the introduction may illustrate this possibility. One should equally note that the aforementioned projections based on 19th-century data also illustrate that the contemporary level of mortality would have been considered unrealistically low from an experiential point of view of the 19th century. For these reasons, we did not apply any lower-bound limits to the projected mortality.

## 5. Conclusion

We start our study by noting that an extrapolative approach to mortality forecasting—though remaining one of the most important practical methods—possesses some drawbacks. In particular, extrapolation of age/sex-specific death rates often results in implausible age/sex patterns of the projected mortality. This feature is in common between the two probably most common extrapolative methods applied to mortality projection by age and sex: to the direct extrapolation of individual age/sex-specific death rates, and to the Lee-Carter method, which does the same indirectly.

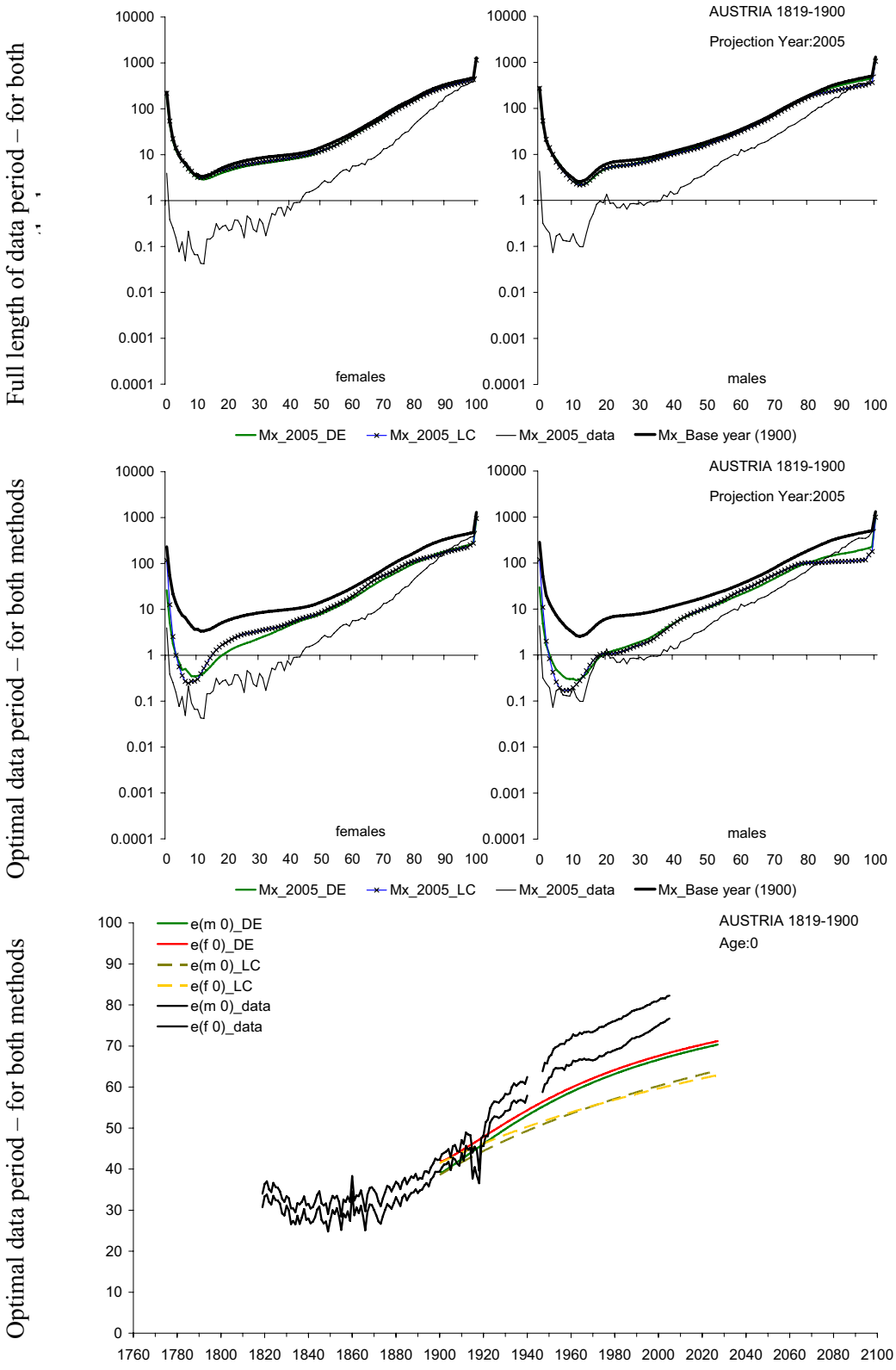
Despite many similarities, however, the direct extrapolation and the Lee-Carter methods are different in a certain way. Most importantly, the methods differ in their assumptions about cross-age/sex autocorrelations in deviations of mortality dynamics from linearity. We examined the relevance of these methods for the reality of mortality dynamics by analyzing autocorrelation patterns in the data from several populations and from different periods of data. Our comparative study of correlations implied that direct extrapolation may be considered a more prominent approach, mainly because it allows flexibility in terms of selecting data periods of different length and also in terms of applying models of different type at different ages. Nonetheless, some populations do show considerable autocorrelations, especially populations with mortality rates significantly affected by epidemics, famines and other causes of death, which is characteristic for the period prior to the modern epidemiological transition. In such cases, the LC approach may indeed be more effective, being applied either to the entire population or to subsets of the age span.

Thus choosing direct extrapolation as the basis for further development of the projection method, we supplement it with additional procedures aimed at improving its short-term projecting efficiency and long-term consistency. In particular, we estimate

optimal age/sex-specific durations of most recent data periods, when linearity of dynamics of log-mortality rates can be assumed. These most recent linear trends are extrapolated into the future. To avoid implausible projected mortality patterns in the long run, however, we also estimate long-run plausible schedules of age/sex-specific rates of mortality decline, toward which mortality dynamics is assumed to be gradually converging. The key assumption used to obtain these schedules is that the long-run rates of mortality decline form a pattern monotonically decreasing with age, the rates of decline for females being higher than those for males. Such an assumption is derived from the general tendency of rates of mortality decline to decrease with age found in empirical observations. Perhaps further research may help either in supporting this assumption or in refining conditions to be imposed on rates of mortality decline in the long run. Other assumptions used in the paper (e.g., minimal duration of the data period, criteria used to detect the optimal duration of the data period, etc.) may also be further improved. Still other improvements could be related to the aforementioned possibilities of introducing cohort effects and of using additional age/sex-specific weighting factors in procedure (15) of fitting the long-run schedule of rates of mortality decline to the recently observed rates.

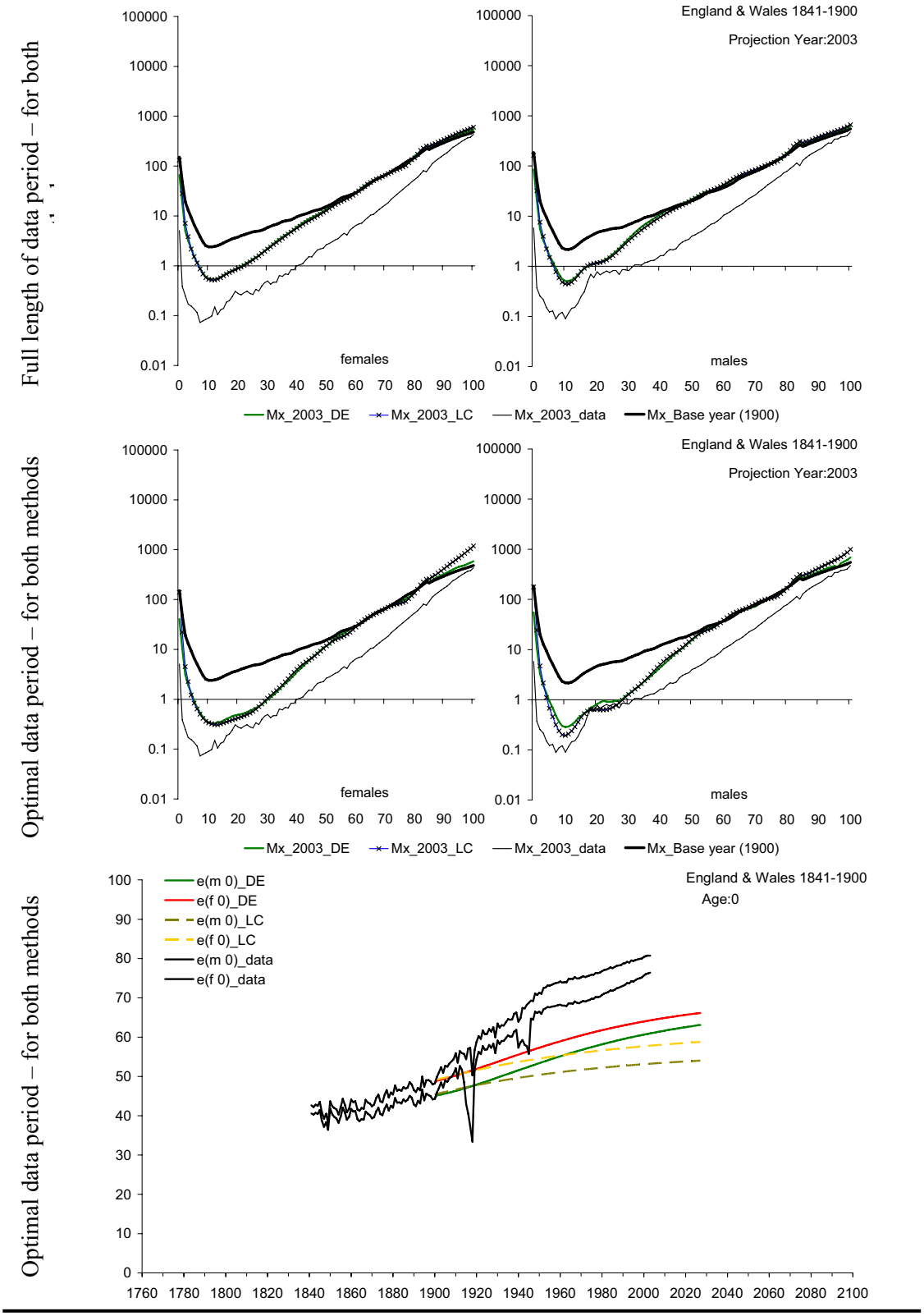
Two major further developments within the framework of the method may be outlined. To begin with, the data, estimates and projections presented for different populations point to the possibility of improving the projection efficiency—especially in the long run—by developing the method in the multi-regional framework. With respect to the LC method, this was successfully done by Li and Lee (2005). For the direct extrapolation method proposed here, a multi-regional approach may be based on assuming all populations to have similar long-run rates of mortality decline (so that an ever-increasing divergence of populations' mortality schedules will be precluded) and, possibly, on assuming that asymptotically, all populations should converge to the same trajectory of mortality dynamics (which, in addition to the long-run rates of decline will also affect the estimates of parameters of convergence to the long-run trend). Another aspect which may briefly be noted concerns introducing uncertainty into the forecast. This may be done either deterministically, by developing variant scenarios of future dynamics, or probabilistically, by developing a stochastic model for mortality dynamics. Both methods may be based on objective measures of uncertainty which may be derived within the framework proposed in the paper. Namely, standard errors of estimates of parameters of age/sex-specific trends, as well as of long-run schedules of rates of mortality decline together with estimates of autocorrelations between model residuals and between deviations of observed rates of mortality decline from long-run estimates may be used for that purpose.

**Figure 8.** Projected age/sex-specific death rates (per 1000, logarithmic scale) and life expectancy at birth since 1900 based on data prior to 1900. Austria.



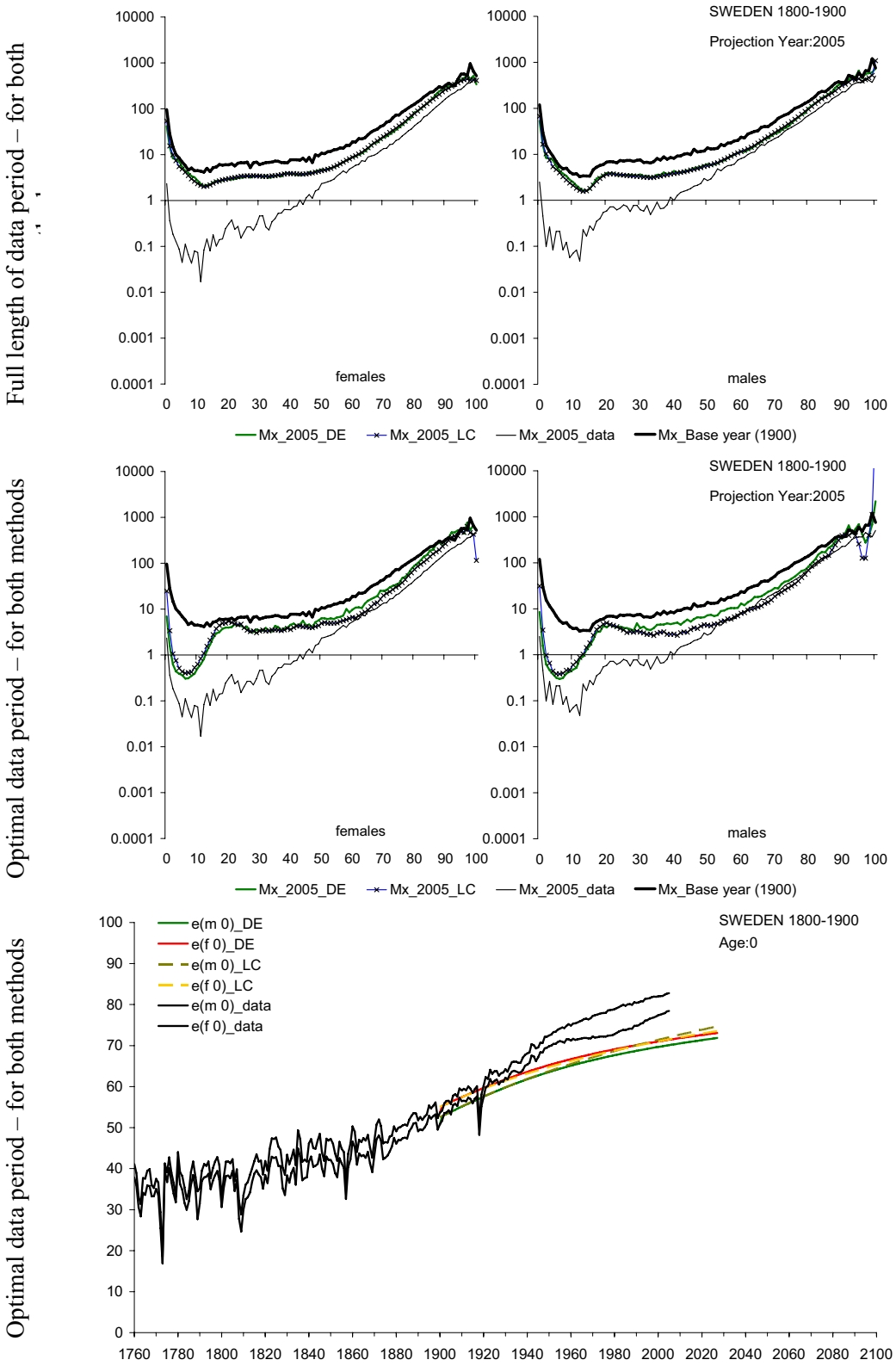

Source: Human Mortality Database, Ediev and Gisser (2007).
Note: 'DE' stands for the 'Direct Extrapolation' method (22); 'LC' stands for the 'Lee-Carter' method.

**Figure 9.** Projected age/sex-specific death rates (per 1000, logarithmic scale) and life expectancy at birth since 1900 based on data prior to 1900. England and Wales.



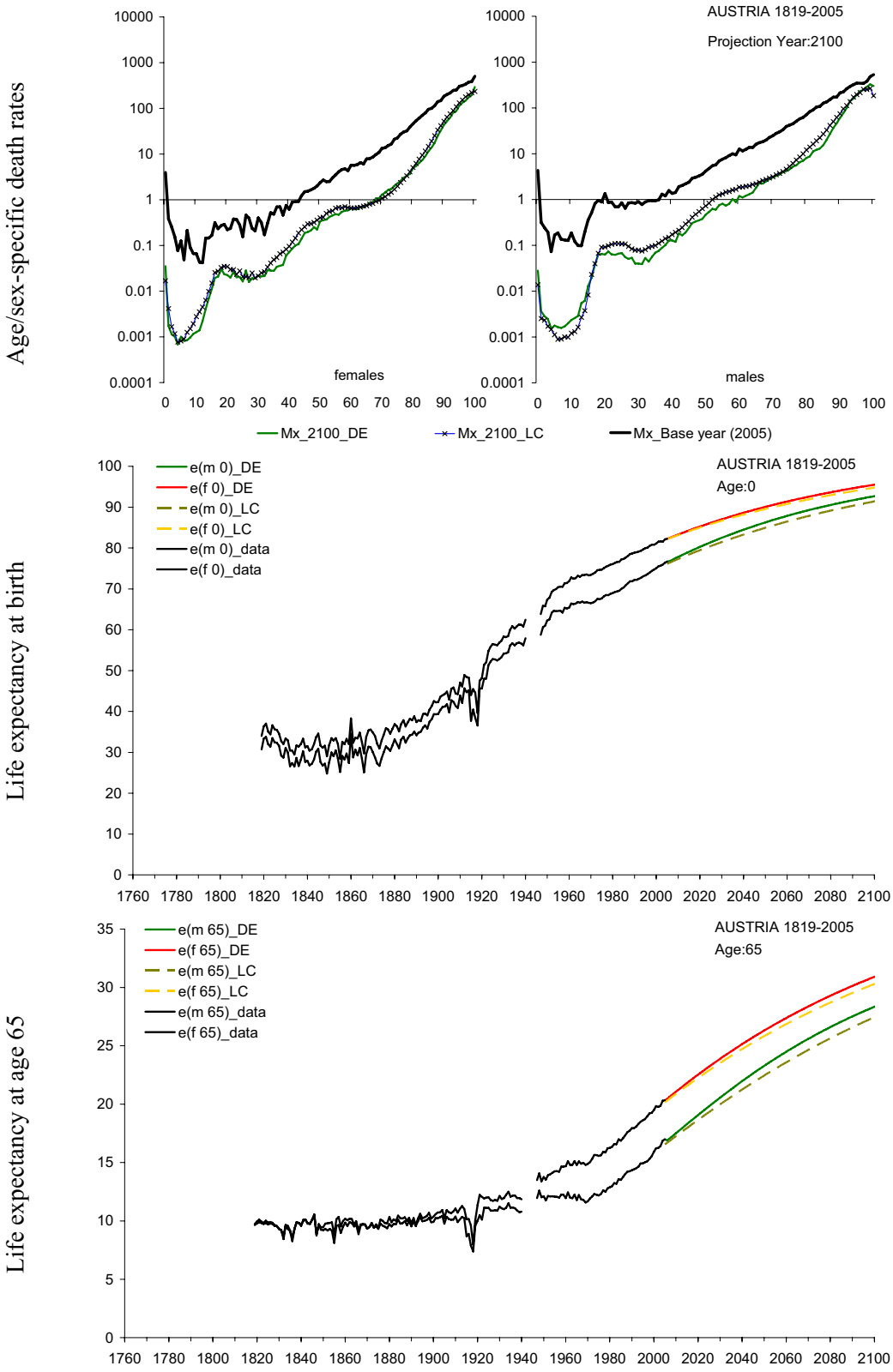

Source: Human Mortality Database, Ediev and Gisser (2007).
Note: 'DE' stands for the 'Direct Extrapolation' method (22); 'LC' stands for the 'Lee-Carter' method.

**Figure 10.** Projected age/sex-specific death rates (per 1000, logarithmic scale) and life expectancy at birth since 1900 based on data prior to 1900. Sweden.



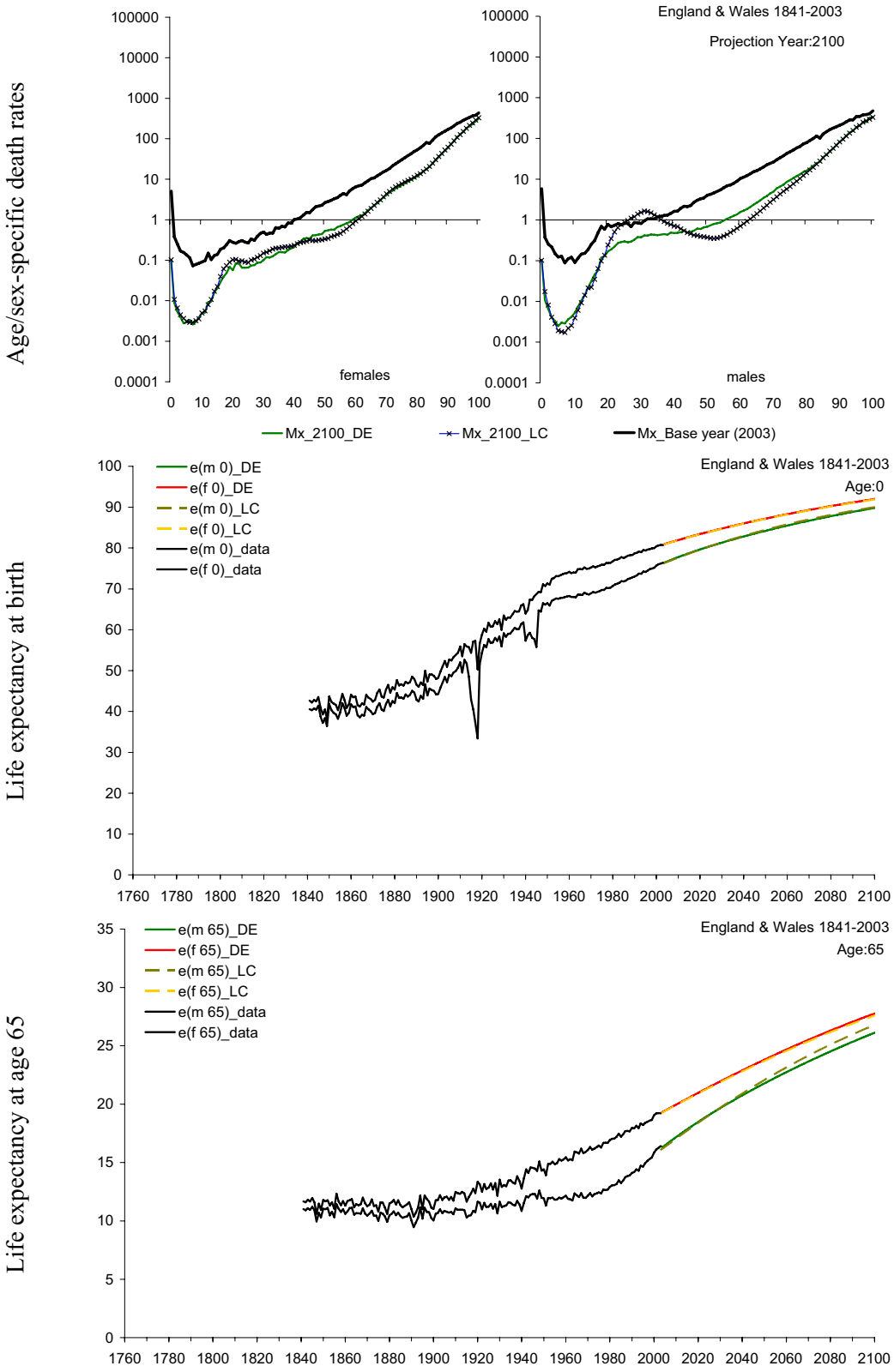

Source: Human Mortality Database, Ediev and Gisser (2007).
Note: 'DE' stands for the 'Direct Extrapolation' method (22); 'LC' stands for the 'Lee-Carter' method.

**Figure 11.** Projected age/sex-specific death rates (per 1000, logarithmic scale) and life expectancies at ages 0 and 65 based on the most recent optimal data periods. Austria.



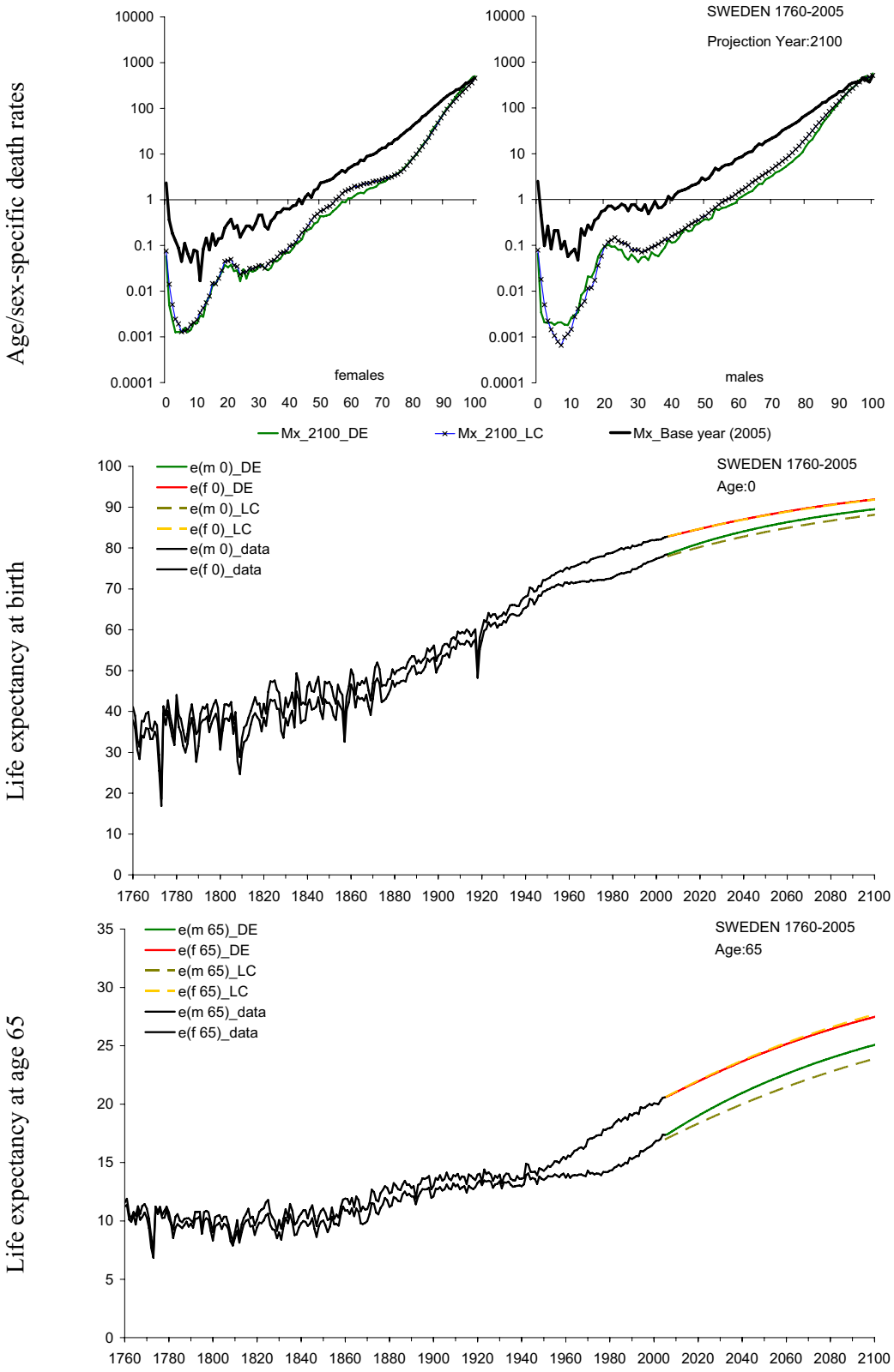

Source: Human Mortality Database, Ediev and Gisser (2007).
Note: 'DE' stands for the 'Direct Extrapolation' method (22); 'LC' stands for the 'Lee-Carter' method.

**Figure 12.** Projected age/sex-specific death rates (per 1000) and life expectancies at ages 0 and 65 based on the most recent optimal data periods. England and Wales.



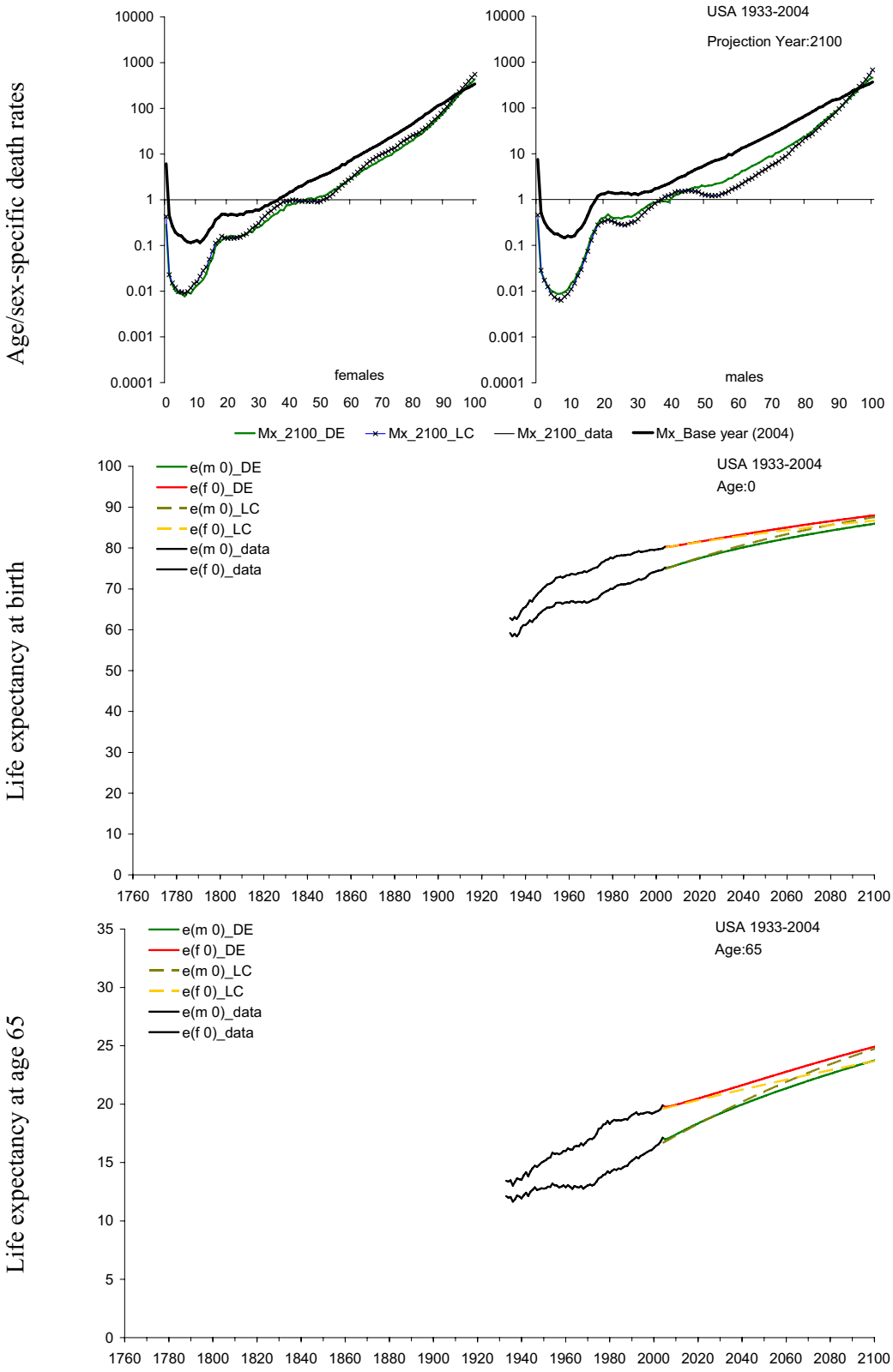

Source: Human Mortality Database, Ediev and Gisser (2007).
Note: 'DE' stands for the 'Direct Extrapolation' method (22); 'LC' stands for the 'Lee-Carter' method.

**Figure 13.** Projected age/sex-specific death rates (per 1000, logarithmic scale) and life expectancies at ages 0 and 65 based on the most recent optimal data periods. Sweden.



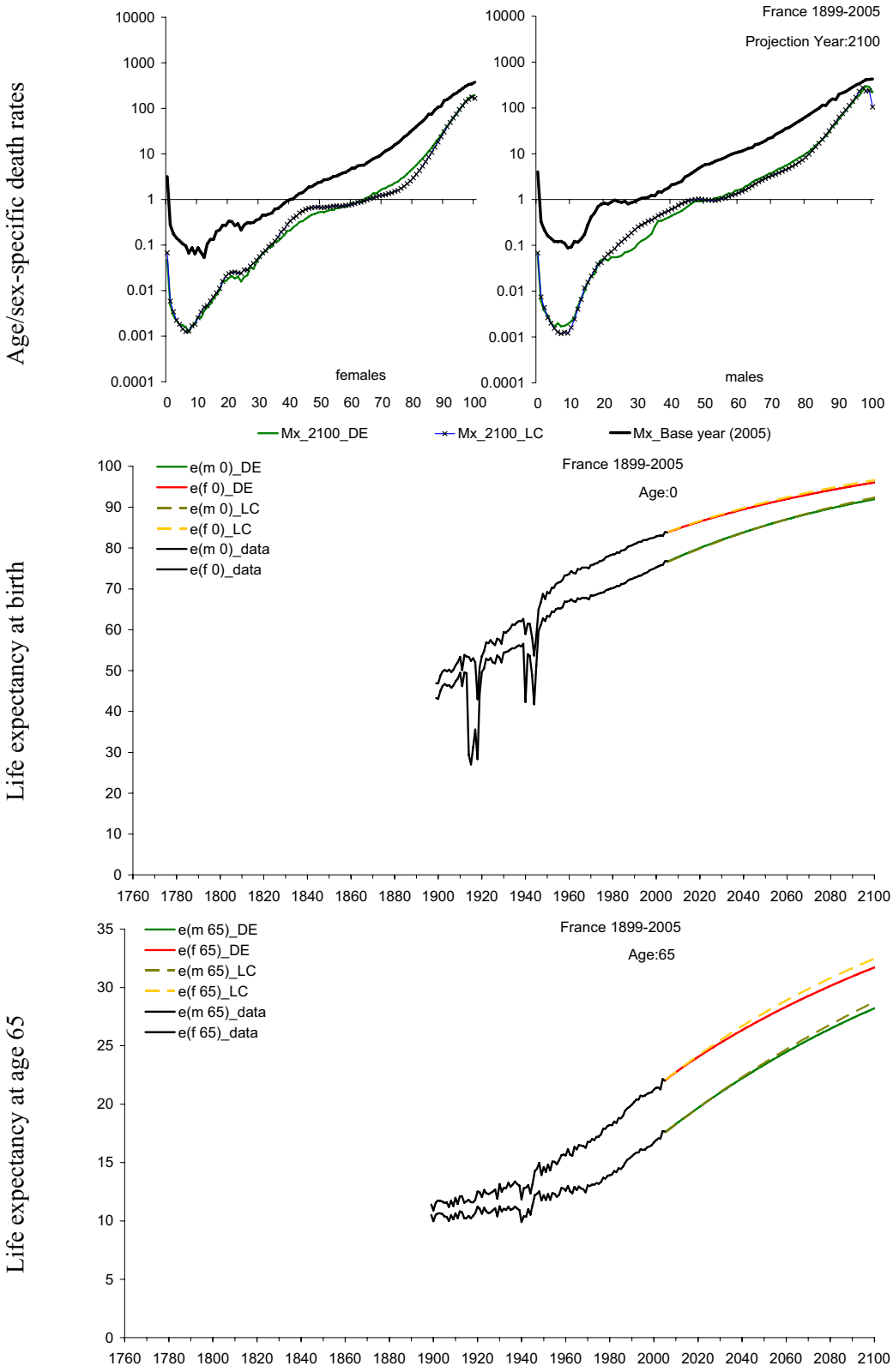

Source: Human Mortality Database, Ediev and Gisser (2007).
Note: 'DE' stands for the 'Direct Extrapolation' method (22); 'LC' stands for the 'Lee-Carter' method.

**Figure 14.** Projected age/sex-specific death rates (per 1000, logarithmic scale) and life expectancies at ages 0 and 65 based on the most recent optimal data periods. US.



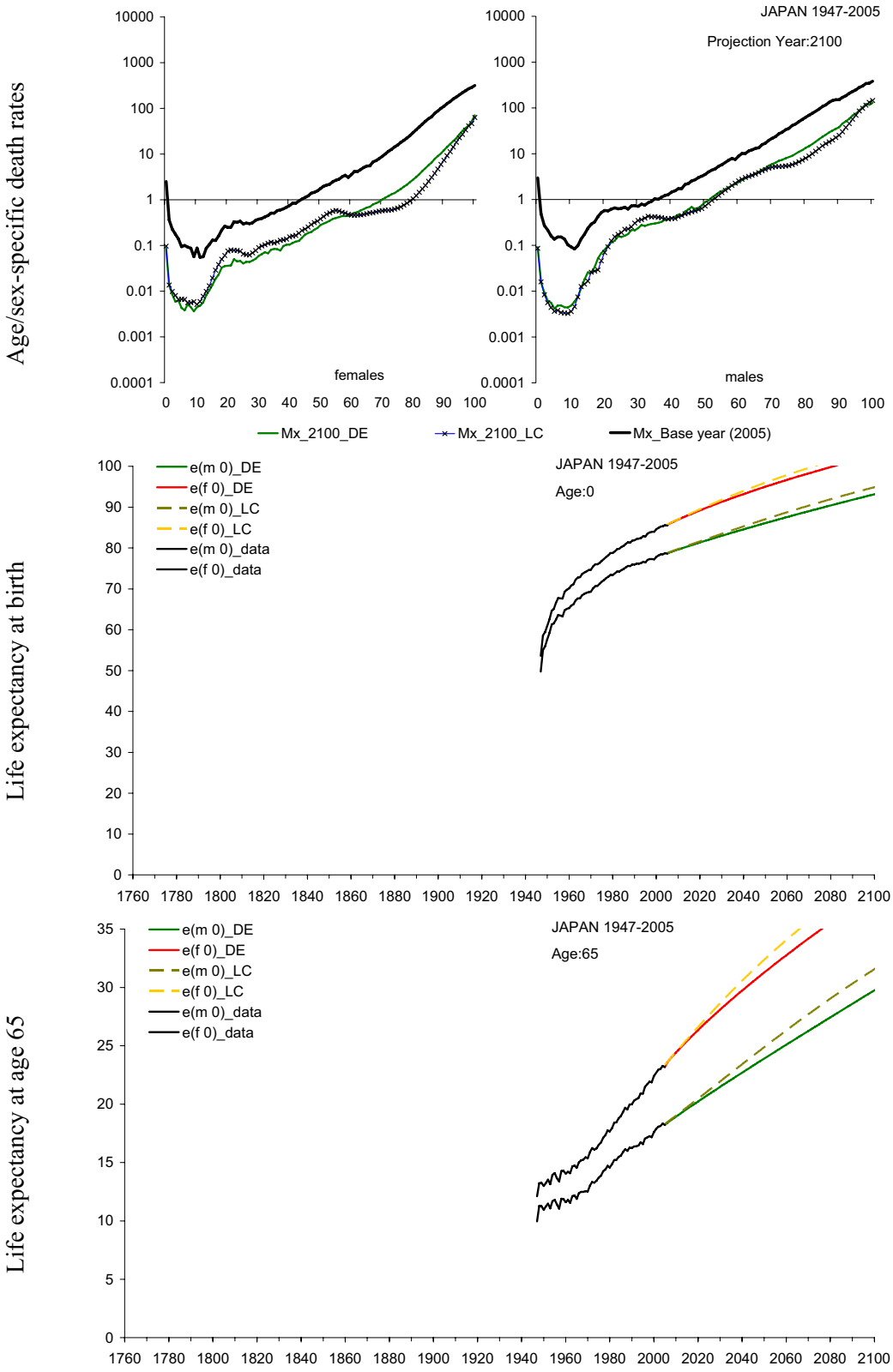

Source: Human Mortality Database, Ediev and Gisser (2007).
Note: 'DE' stands for the 'Direct Extrapolation' method (22); 'LC' stands for the 'Lee-Carter' method.

**Figure 15.** Projected age/sex-specific death rates (per 1000, logarithmic scale) and life expectancies at ages 0 and 65 based on the most recent optimal data periods. France.



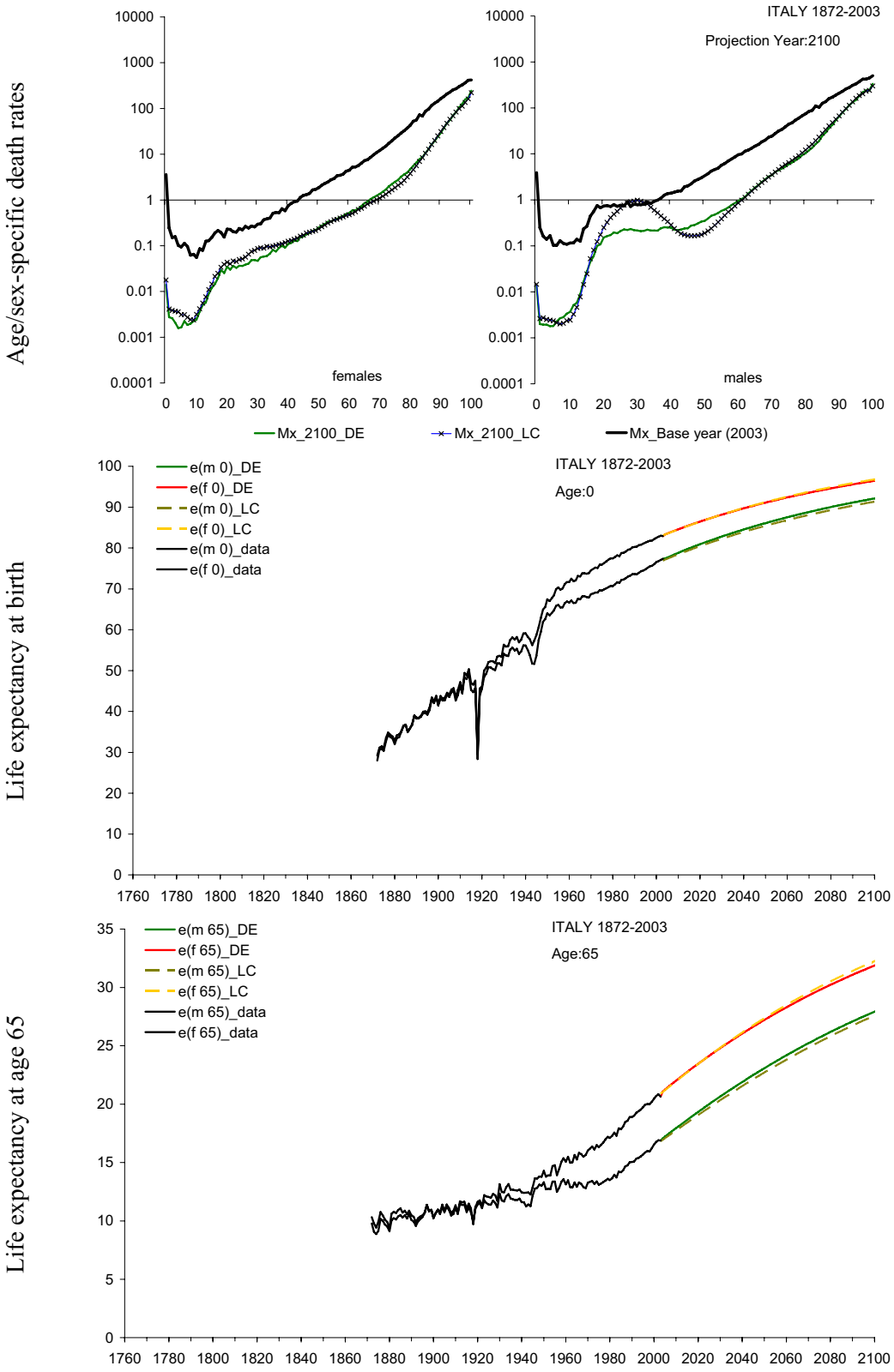

Source: Human Mortality Database, Ediev and Gisser (2007).
Note: 'DE' stands for the 'Direct Extrapolation' method (22); 'LC' stands for the 'Lee-Carter' method.

**Figure 16.** Projected age/sex-specific death rates (per 1000, logarithmic scale) and life expectancies at ages 0 and 65 based on the most recent optimal data periods. Japan.



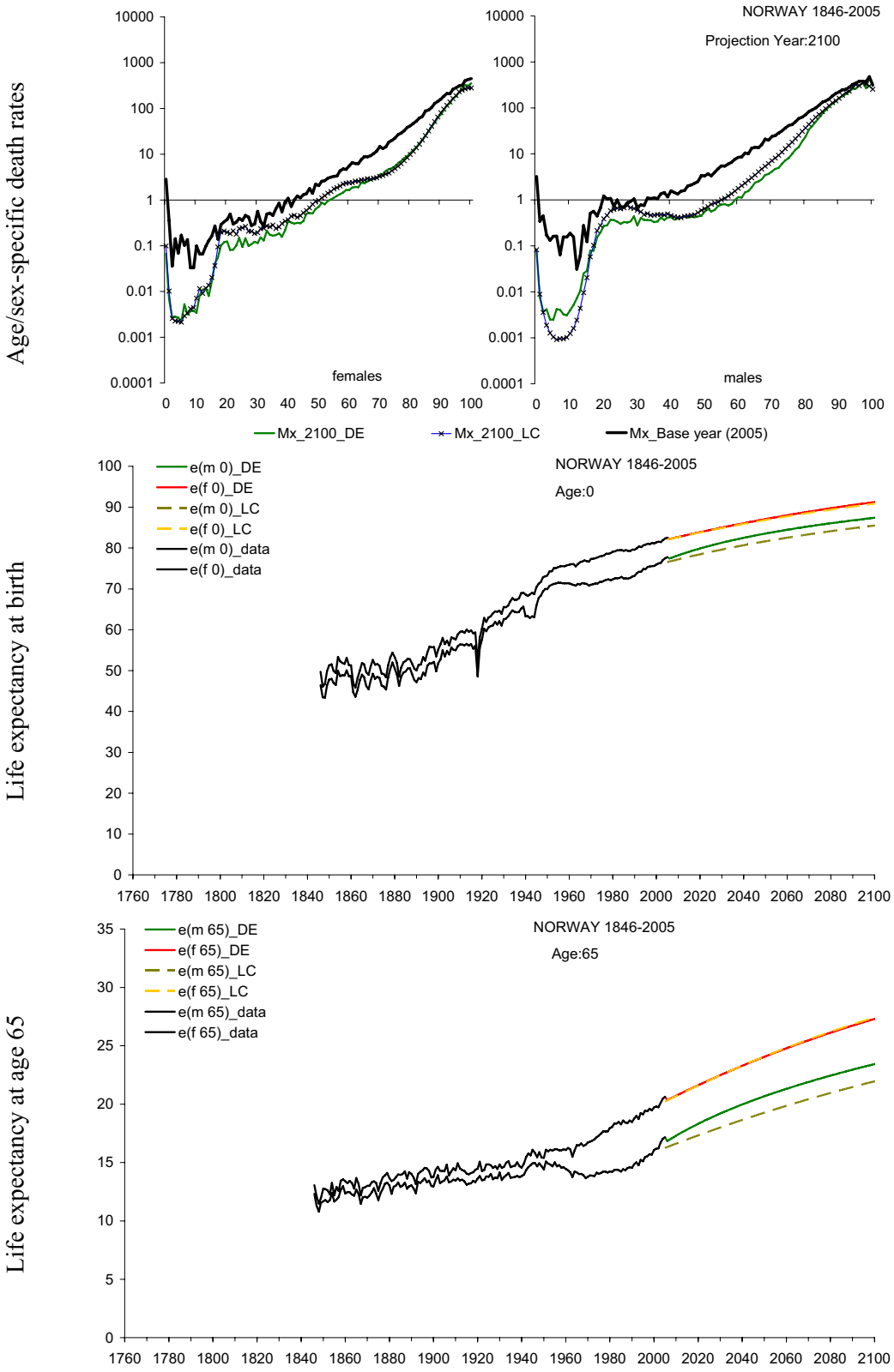

Source: Human Mortality Database, Ediev and Gisser (2007).
Note: 'DE' stands for the 'Direct Extrapolation' method (22); 'LC' stands for the 'Lee-Carter' method.

**Figure 17.** Projected age/sex-specific death rates (per 1000, logarithmic scale) and life expectancies at ages 0 and 65 based on the most recent optimal data periods. Italy.




Source: Human Mortality Database, Ediev and Gisser (2007).
Note: 'DE' stands for the 'Direct Extrapolation' method (22); 'LC' stands for the 'Lee-Carter' method.

**Figure 18.** Projected age/sex-specific death rates (per 1000, logarithmic scale) and life expectancies at ages 0 and 65 based on the most recent optimal data periods. Norway.




Source: Human Mortality Database, Ediev and Gisser (2007).
Note: 'DE' stands for the 'Direct Extrapolation' method (22); 'LC' stands for the 'Lee-Carter' method.

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# Appendix

**Derivation of non-biased estimates for dispersions of the autocorrelated error components in Eq. (3)**

For the sake of simplicity, everywhere in the Appendix we suppose the time variable standardized in such a way that $\bar{t} = 0$.

Preliminary results.

For any vectors $\mathbf{x}^T = (x_1, x_2, ,..., x_n)$, $\mathbf{y}^T = (y_1, y_2, ,..., y_n)$:

$$\mathbf{x}^T C^{-1} \mathbf{y} = (1 + 2c\rho)\sum_{i=1}^{n} x_i y_i - c\rho(x_n y_n + x_1 y_1) - c\sum_{i=1}^{n-1}(x_{i+1} y_i + x_i y_{i+1}) =$$

$$= (1 - 2c(1-\rho))\sum_{i=1}^{n} x_i y_i + c((2-\rho)x_n y_n - \rho x_1 y_1) - c\sum_{i=1}^{n-1}((x_{i+1} - x_i) y_i + x_i (y_{i+1} - y_i))$$

$$= \frac{1-\rho}{1+\rho}\sum_{i=1}^{n} x_i y_i + \frac{\rho}{1-\rho^2}((2-\rho)x_n y_n - \rho x_1 y_1) - \frac{\rho}{1-\rho^2}\sum_{i=1}^{n-1}((x_{i+1} - x_i) y_i + x_i (y_{i+1} - y_i)) \quad \text{(A1)}$$

Using this result, it may be shown that

$$X^T C^{-1} X = \begin{pmatrix} n - (n-1)\dfrac{2\rho}{1+\rho} & 0 \\ 0 & n\overline{t^2}\left(1 - \dfrac{2\rho}{1+\rho}\right) + \dfrac{(n-1)^2}{2}\dfrac{\rho}{1+\rho} + (n-1)\dfrac{\rho}{1-\rho^2} \end{pmatrix} =$$

$$= \frac{n}{1+\rho}\begin{pmatrix} 1 - \rho + 2\rho/n & 0 \\ 0 & \overline{t^2}(1-\rho) + \dfrac{\rho(n-1)^2}{2n} + \dfrac{n-1}{n}\dfrac{\rho}{1-\rho} \end{pmatrix}. \quad \text{(A2)}$$

$$\left(X^T C^{-1} X\right)^{-1} = \frac{(1+\rho)}{n}\begin{pmatrix} \dfrac{1}{(1-\rho) + 2\rho/n} & 0 \\ 0 & \dfrac{1}{\overline{t^2}(1-\rho) + \dfrac{\rho(n-1)^2}{2n} + \dfrac{n-1}{n}\dfrac{\rho}{1-\rho}} \end{pmatrix} \quad \text{(A3)}$$

Consider the following (biased) estimate for errors' dispersion $\sigma^2 = E(\varepsilon_i^2)$, $i = \overline{1,n}$:

$$\hat{\sigma}^2 = \frac{\hat{E}^T C^{-1} \hat{E}}{n} = \frac{1}{n}\left(\hat{Y} - Y\right)^T C^{-1}\left(\hat{Y} - Y\right) = \frac{1}{n}\left(X\hat{\mathbf{a}} - (X\mathbf{a} + \boldsymbol{\varepsilon})\right)^T C^{-1}\left(X\hat{\mathbf{a}} - (X\mathbf{a} + \boldsymbol{\varepsilon})\right) =$$

$$= \frac{1}{n}\left(X\left(X^T C^{-1} X\right)^{-1} X^T C^{-1} Y - (X\mathbf{a} + \boldsymbol{\varepsilon})\right)^T C^{-1}\left(X\left(X^T C^{-1} X\right)^{-1} X^T C^{-1} Y - (X\mathbf{a} + \boldsymbol{\varepsilon})\right) =$$

$$= \frac{1}{n}\left(X\left(X^T C^{-1} X\right)^{-1} X^T C^{-1}(X\mathbf{a} + \boldsymbol{\varepsilon}) - (X\mathbf{a} + \boldsymbol{\varepsilon})\right)^T C^{-1}\left(X\left(X^T C^{-1} X\right)^{-1} X^T C^{-1}(X\mathbf{a} + \boldsymbol{\varepsilon}) - (X\mathbf{a} + \boldsymbol{\varepsilon})\right) =$$

$$= \frac{1}{n}\left(X\left(X^T C^{-1} X\right)^{-1} X^T C^{-1}\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}\right)^T C^{-1}\left(X\left(X^T C^{-1} X\right)^{-1} X^T C^{-1}\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}\right) =$$

$$= \frac{1}{n}\boldsymbol{\varepsilon}^T C^{-1}\boldsymbol{\varepsilon} - \frac{1}{n}\boldsymbol{\varepsilon}^T C^{-1} X\left(X^T C^{-1} X\right)^{-1} X^T C^{-1}\boldsymbol{\varepsilon}\,. \tag{A4}$$

The first summand in (A4) is computed straightforwardly from (A1):

$$\frac{1}{n}\boldsymbol{\varepsilon}^T C^{-1}\boldsymbol{\varepsilon} = \overline{\varepsilon^2}\frac{1-\rho}{1+\rho} + \frac{\rho}{(1-\rho^2)n}\left((2-\rho)\varepsilon_n{}^2 - \rho\varepsilon_1{}^2\right) - \frac{2\rho}{(1-\rho^2)n}\sum_{i=1}^{n-1}\varepsilon_i(\varepsilon_{i+1}-\varepsilon_i) \tag{A5}$$

The expected value of this expression is further simplified

$$E\left(\frac{1}{n}\boldsymbol{\varepsilon}^T C^{-1}\boldsymbol{\varepsilon}\right) = E\left(\overline{\varepsilon^2}\right)\frac{1-\rho}{1+\rho} + \frac{\rho}{(1-\rho^2)n}E\left((2-\rho)\varepsilon_n{}^2 - \rho\varepsilon_1{}^2\right) - \frac{2\rho}{(1-\rho^2)n}E\left(\sum_{i=1}^{n-1}\varepsilon_i(\varepsilon_{i+1}-\varepsilon_i)\right) =$$

$$= \sigma^2\frac{1-\rho}{1+\rho} + \frac{\rho}{(1-\rho^2)n}\sigma^2\left((2-\rho)-\rho\right) - \frac{2\rho}{(1-\rho^2)n}\sigma^2\sum_{i=1}^{n-1}(\rho-1) =$$

$$= \sigma^2\frac{1-\rho}{1+\rho} + \frac{2\sigma^2\rho}{(1+\rho)n} + \frac{2\sigma^2\rho(n-1)}{(1+\rho)n} = \sigma^2\,. \tag{A6}$$

To simplify the second summand in (A4), e note, first, that

$$\boldsymbol{\varepsilon}^T C^{-1} X = n\frac{1-\rho}{1+\rho}\left(\begin{array}{cc} \bar{\varepsilon} + \dfrac{\rho(\varepsilon_n+\varepsilon_1)}{n(1-\rho)} & \overline{t\varepsilon} + \rho\dfrac{n(1-\rho)+1+\rho}{2(1-\rho)^2 n}(\varepsilon_n-\varepsilon_1)\end{array}\right) \tag{A7}$$

Hence,

$$\frac{1}{n}\boldsymbol{\varepsilon}^T C^{-1} X\left(X^T C^{-1} X\right)^{-1} X^T C^{-1}\boldsymbol{\varepsilon} = \frac{1}{n}n\frac{1-\rho}{1+\rho}\left(\begin{array}{cc} \bar{\varepsilon} + \dfrac{\rho(\varepsilon_n+\varepsilon_1)}{n(1-\rho)} & \overline{t\varepsilon} + \rho\dfrac{n(1-\rho)+1+\rho}{2(1-\rho)^2 n}(\varepsilon_n-\varepsilon_1)\end{array}\right)\times$$

$$\times\frac{(1+\rho)}{n}\left(\begin{array}{cc} \dfrac{1}{(1-\rho)+2\rho/n} & 0 \\ 0 & \dfrac{1}{\overline{t^2}(1-\rho)+\dfrac{\rho(n-1)^2}{2n}+\dfrac{n-1}{n}\dfrac{\rho}{1-\rho}}\end{array}\right)\times$$

$$\times n\frac{1-\rho}{1+\rho}\left(\begin{array}{c} \bar{\varepsilon} + \dfrac{\rho(\varepsilon_n+\varepsilon_1)}{n(1-\rho)} \\ \overline{t\varepsilon} + \rho\dfrac{n(1-\rho)+1+\rho}{2(1-\rho)^2 n}(\varepsilon_n-\varepsilon_1)\end{array}\right) =$$

$$= \frac{(1-\rho)^2}{1+\rho}\left(\begin{array}{c} \dfrac{\left(\bar{\varepsilon} + \dfrac{\rho(\varepsilon_n+\varepsilon_1)}{n(1-\rho)}\right)^2}{(1-\rho)+2\rho/n} + \dfrac{\left(\overline{t\varepsilon} + \rho\dfrac{n(1-\rho)+1+\rho}{2(1-\rho)^2 n}(\varepsilon_n-\varepsilon_1)\right)^2}{\overline{t^2}(1-\rho)+\dfrac{\rho(n-1)^2}{2n}+\dfrac{n-1}{n}\dfrac{\rho}{1-\rho}}\end{array}\right). \tag{A8}$$

To derive the expected value for (A8), we note that

$$E\left(\bar{\varepsilon}^2\right)=\frac{1}{n^2}E\left(\sum_i \varepsilon_i \sum_j \varepsilon_j\right)=\frac{\sigma^2}{n}+\frac{\sigma^2}{n^2}\sum_i\sum_{j\neq i}\rho^{|i-j|}=\frac{\sigma^2}{n}+\frac{2\sigma^2}{n^2}\sum_{i>1}\sum_{j=1}^{i-1}\rho^{i-j}=$$
$$=\frac{\sigma^2}{n}+\frac{2\sigma^2}{n^2}\sum_{i>1}\rho^i\frac{\rho^{-i+1}-1}{1-\rho}=\frac{\sigma^2}{n}+\frac{2\sigma^2}{n^2}\sum_{i>1}\frac{\rho-\rho^i}{1-\rho}=$$
$$=\frac{\sigma^2}{n}+\frac{2\sigma^2\rho(n-1)}{n^2(1-\rho)}-\frac{2\sigma^2\rho^2\left(1-\rho^{n-1}\right)}{n^2(1-\rho)^2}. \qquad \text{(A9)}$$

$$E\left(\bar{\varepsilon}(\varepsilon_n+\varepsilon_1)\right)=E\left(\frac{\sum_i \varepsilon_i(\varepsilon_n+\varepsilon_1)}{n}\right)=\frac{\sigma^2\sum_i\left(\rho^{n-i}+\rho^{i-1}\right)}{n}=\frac{2\sigma^2\sum_i\rho^{i-1}}{n}=$$
$$=\frac{2\sigma^2\left(1-\rho^n\right)}{n(1-\rho)}. \qquad \text{(A10)}$$
$$E\left((\varepsilon_n+\varepsilon_1)^2\right)=2\sigma^2\left(1+\rho^{n-1}\right). \qquad \text{(A11)}$$
$$E\left(\overline{t\varepsilon}^2\right)=\frac{1}{n^2}E\left(\sum_i t_i\varepsilon_i\sum_j t_i\varepsilon_j\right)=\frac{\sigma^2\overline{t^2}}{n}+\frac{2\sigma^2}{n^2}\sum_i t_i\sum_{j<i}t_j\rho^{i-j}. \qquad \text{(A12)}$$
$$E\left(\overline{t\varepsilon}(\varepsilon_n-\varepsilon_1)\right)=\frac{\sigma^2}{n}\sum_i t_i\left(\rho^{n-i}-\rho^{i-1}\right)=\frac{2\sigma^2}{n}\sum_i t_i\rho^{n-i}=\frac{2\sigma^2\rho^n}{n}\sum_i t_i\rho^{-i}. \qquad \text{(A13)}$$
$$E\left((\varepsilon_n-\varepsilon_1)^2\right)=2\sigma^2\left(1-\rho^{n-1}\right). \qquad \text{(A14)}$$

Expanding the expressions in parenthesis in (A8) and substituting (A9)-(A14), we get from (A4):

$$E\left(\hat{\sigma}^2\right)=\sigma^2-E\left(\frac{1}{n}\boldsymbol{\varepsilon}^T C^{-1}X\left(X^TC^{-1}X\right)^{-1}X^TC^{-1}\boldsymbol{\varepsilon}\right)=\sigma^2-$$
$$-\frac{(1-\rho)^2}{1+\rho}\left(\frac{E\left(\bar{\varepsilon}^2\right)+\frac{2\rho}{n(1-\rho)}E\left(\bar{\varepsilon}(\varepsilon_n+\varepsilon_1)\right)+\left(\frac{\rho}{n(1-\rho)}\right)^2E\left((\varepsilon_n+\varepsilon_1)^2\right)}{(1-\rho)+2\rho/n}+\right.$$
$$\left.+\frac{E\left(\overline{t\varepsilon}^2\right)+2\rho\frac{n(1-\rho)+1+\rho}{2(1-\rho)^2n}E\left(\overline{t\varepsilon}(\varepsilon_n-\varepsilon_1)\right)+\left(\rho\frac{n(1-\rho)+1+\rho}{2(1-\rho)^2n}\right)^2E\left((\varepsilon_n-\varepsilon_1)^2\right)}{\overline{t^2}(1-\rho)+\frac{\rho(n-1)^2}{2n}+\frac{n-1}{n}\frac{\rho}{1-\rho}}\right)=$$
$$=\sigma^2-\frac{(1-\rho)^2}{(1+\rho)((1-\rho)+2\rho/n)}\left\{\frac{\sigma^2}{n}+\frac{2\sigma^2\rho(n-1)}{n^2(1-\rho)}-\frac{2\sigma^2\rho^2\left(1-\rho^{n-1}\right)}{n^2(1-\rho)^2}+\right.$$

$$+\frac{2\rho}{n(1-\rho)}\frac{2\sigma^2(1-\rho^n)}{n(1-\rho)}+\left(\frac{\rho}{n(1-\rho)}\right)^2 2\sigma^2(1+\rho^{n-1})\Bigg\}-$$

$$-\frac{(1-\rho)^2}{1+\rho}\frac{1}{\overline{t^2}(1-\rho)+\frac{\rho(n-1)^2}{2n}+\frac{n-1}{n}\frac{\rho}{1-\rho}}\Bigg[\frac{\sigma^2\overline{t^2}}{n}+\frac{2\sigma^2}{n^2}\sum_i t_i\sum_{j<i}t_j\rho^{i-j}+$$

$$+2\rho\frac{n(1-\rho)+1+\rho}{2(1-\rho)^2 n}\frac{2\sigma^2\rho^n}{n}\sum_i t_i\rho^{-i}+\left(\rho\frac{n(1-\rho)+1+\rho}{2(1-\rho)^2 n}\right)^2 2\sigma^2(1-\rho^{n-1})\Bigg]=$$

$$=\sigma^2-\frac{(1-\rho)^2}{(1+\rho)((1-\rho)+2\rho/n)}\left\{\frac{\sigma^2}{n}+\frac{2\sigma^2\rho(n-1)}{n^2(1-\rho)}+\frac{4\sigma^2\rho}{n^2(1-\rho)^2}\right\}-$$

$$-\frac{(1-\rho)^2}{1+\rho}\frac{1}{\overline{t^2}(1-\rho)+\frac{\rho(n-1)^2}{2n}+\frac{n-1}{n}\frac{\rho}{1-\rho}}\Bigg[\frac{\sigma^2\overline{t^2}}{n}+\frac{2\sigma^2}{n^2}\sum_i t_i\sum_{j<i}t_j\rho^{i-j}+$$

$$+2\rho\frac{n(1-\rho)+1+\rho}{2(1-\rho)^2 n}\frac{2\sigma^2\rho^n}{n}\sum_i t_i\rho^{-i}+\left(\rho\frac{n(1-\rho)+1+\rho}{2(1-\rho)^2 n}\right)^2 2\sigma^2(1-\rho^{n-1})\Bigg]=$$

$$=\sigma^2-\frac{\sigma^2}{n}\frac{1}{(1+\rho)((1-\rho)+2\rho/n)}\left\{(1-\rho)^2+\frac{2\rho(1-\rho)(n-1)}{n}+\frac{4\rho}{n}\right\}-$$

$$-\frac{\sigma^2}{n}\frac{1}{1+\rho}\frac{1}{\overline{t^2}(1-\rho)^2+\frac{\rho(1-\rho)(n-1)^2}{2n}+\frac{n-1}{n}\rho}\Bigg[\overline{t^2}(1-\rho)^3+\frac{2}{n}(1-\rho)^3\sum_i t_i\sum_{j<i}t_j\rho^{i-j}+$$

$$+2\rho(1-\rho)\frac{n(1-\rho)+1+\rho}{n}\sum_i t_i\rho^{n-i}+(n(1-\rho)+1+\rho)^2\frac{\rho^2(1+\rho+\rho^2+\ldots+\rho^{n-2})}{2n}\Bigg]=$$

$$=\sigma^2-\frac{\sigma^2}{n}-\frac{\sigma^2}{n}=\frac{\sigma^2(n-2)}{n}. \qquad \text{(A15)}$$

Hence, the unbiased estimate is given by the following formulae:

$$\hat{\sigma}_\varepsilon{}^2=\frac{\hat{E}^T C^{-1}\hat{E}}{n-2}=\frac{1}{(n-2)(1-\rho^2)}\left(\sum_{i=1}^{n}e_i{}^2+\rho^2\sum_{i=2}^{n-1}e_i{}^2-2\rho\sum_{i=1}^{n-1}e_i e_{i+1}\right). \qquad \text{(A16)}$$

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