On the Theory of Distortions of Period Estimates of the Quantum Caused by the Tempo Changes

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ABSTRACT

This paper presents a general theory of tempo and quantum of life events based on the notion of infinitesimally short life stages successively covering the life span of every cohort. It is shown that tempo distortions are of a purely geometrical nature which is not linked to the type of events, to cohort or period perspective nor to the type of rates describing the demographic process under study. Basically, the distortions appear as a consequence of different exposure of real birth cohorts and of synthetic cohorts to the same life stages. The paper provides general adjustment formulas for distortions at individual ages, which, being integrated over the life span, provide adjustments for integral demographic indicators. Our results suggest life stages to be defined from sound demographic, sociological, economic and other considerations. However, we also provide several approaches to defining the life stages and to tempo adjustments based on cumulated proportions of life cycle events in cohorts and in periods. The first approach is shown to generalise Ryder's translation theory, and we provide a general translation equation free of the assumptions used in the literature. The second approach, based on period proportions, generalises Bongaarts and Feeney's method of estimating the dominant period factor of densities of life cycle events. We show that Bongaarts and Feeney's method may be justified without referring to the shifting hypothesis. We also show that there is in fact an indefinite number of internally consistent adjustments, of which the Bongaarts-Feeney, Kohler-Philipov and other adjustments are particular examples. We generalise some of the results by Kohler and Philipov without using their assumption about cumulated postponement and present a general approach to address effects of the variance and of moments of higher order. For any of the internally consistent adjustments, we present a method of deriving full age patterns of the adjusted rates in addition to the traditionally computed total rates. The paper is supplemented by empirical illustrations.

European Demographic Research Papers are working papers that deal with all-European issues or with issues that are important to a large number of countries. All contributions have received only limited review.

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INTRODUCTION

Since the classical works were written by Ryder (1951, 1956, 1959, 1964, 1980), who advocated the cohort approach and proposed the translation equations to estimate cohort completed fertility from the period TFR, and by Bongaarts and Feeney (1998, 2002, 2003, 2006, 2008), who developed an entirely period approach, the topic of distortions of period estimates of quantum due to changes of the tempo of life cycle events has attracted wide attention in both methodological works (e.g. Kohler and Philipov 2001; Yi and Land 2001; Wachter 2005; Guillot 2006; Goldstein 2006) and applied ones (e.g. Sobotka 2004; Winkler-Dworak and Engelhardt 2004; Luy 2006)—see also the references in the works cited above as well as in Bongaarts and Feeney 2006. The phenomenon was shown to apply not only to traditional fertility measures, but also to mortality and marriage measures. Bongaarts and Feeney (2006) advocated the relevance of the concept to any kind of life cycle events. Despite the considerable development in methodology and the experience gained in applications, however, the topic remains controversial. In particular, questions related to whether it is justifiable and how to extend the concept beyond the traditional area of fertility indicators, or what improvements could be made to the formulas by Ryder and Bongaarts-Feeney (which were derived from simplified assumptions) are still not resolved.

This paper presents a new approach to the problem based on a new definition of tempo and quantum. Traditionally, quantum is introduced as an integral indicator of the number of demographic events over a given period or cohort. However, this creates methodological problems with respect to the theory of tempo changes, as the processes of postponement and advancement are of local nature: a change in the personal schedule of a person’s life cycle events around, say, age 18 does not necessarily lead to similar changes at a considerably higher age. Indeed, some of the traditional assumptions employed in the literature may be interpreted as ways of resolving this integral quantum-local tempo distortions controversy (e.g. the hypothesis of the age schedule of rates being shifted without changing its shape). Kohler and Philipov (2001) settled the controversy by introducing a cumulative, i.e. integral measure of postponement. In the mortality context, Feeney (2006) proposed a similar measure of increments to life. Second, the definition of quantum as the integral indicator of a given demographic phenomenon requires defining the demographic model. This brings up a host of questions: what kind of rates are better reflections of demographic dynamics, how to derive integral indicators from individual rates, how to extend the theory traditionally developed for fertility models to the case of mortality models, what—period or cohort?—indices to build the theory on, etc. (see e.g. the discussion of some related problems in Van Imhoff (2001).)

Here, an approach is proposed that is based on a *local* definition of *tempo*, which allows defining quantum *in local terms* as well and also locally addressing the problem of tempo-induced distortions in the period estimates of the quantum. In this framework, it can be demonstrated that many methodological questions like those mentioned above may be separated from the phenomenon of tempo distortions and should therefore be addressed separately, without complicating the theory of the tempo distortions as such.

Let us consider the life course to be split into a set of *successive* stages, i.e. stages that may not overlap. One may consider, for instance, mean age at physiological maturation, age at graduation from the university, age at first ever taking medical treatment of ageing-related illness, etc., to mark the transition from one life stage to another. Let us imagine, further, that it is possible to define the structure of life stages so precisely that the (indefinite number of) infinitesimally small life stages or, equivalently, ages marking the transition from one stage to another fill the entire age axis. (One may consider birth cohorts moving through a continuous set of physiological, psychological, economic, social, etc., etc. states). The set of the life stages or of ages marking the transitions from stage to stage constitutes what I call the *tempo*. When the positioning of life stages along the age axis changes (e.g. when the average age at a certain level of physiological ageing increases), I consider this as *change in the tempo* of the life cycle. Indeed, the definition of life stages must depend on the focus of the study: births, deaths and other life cycle events may be properly studied under different subdivisions of the life course. (Further down in the paper, I will propose some simplified definitions of life stages based on the distribution by age of the life cycle events under study, which generalise the theories known from the literature.)

The tempo defined, we may introduce the quantum in local terms. Let x_1 and x_2 be two successive ages, at which a representative person[1] reaches some stage in his or her life course and, subsequently, leaves it to the next stage. The number of demographic events under study between these two ages I call *the quantum* of the stage the person stayed between ages x_1 and x_2. Traditionally, demographers are interested in what I call the *integral quantum* which characterises the number of life cycle events over the entire life span. In the theory presented here, the integral quantum may be obtained by integrating the local quantum over the life span. In addition to this traditional usage, the quantum may also be of interest when it is estimated for certain life stages only. In studying causes of death, for instance, it might be of relevance to study how many deaths occur after reaching one medical condition and before reaching another one. Epidemiologists may be interested to know how many contacts a person is engaged in between getting infected and being quarantined from the general population. In the context of housing and population policies one might be interested to know

[1] We do not consider here the problem of heterogeneity of the population. As long as life stages are defined, heterogeneity of the population does not affect our findings. Heterogeneity, however, may affect the definition of life stages, as people may move through life stages at different ages or may share different sets of life stages. Traditional methods of overcoming such problems may be suggested here as well: defining life stages based on average or most characteristic life cycle patterns or sharpening the analysis by subdividing the population into subgroups with significantly different behaviours (e.g. considering separately births of different orders as has become common in fertility tempo studies). Also, results shown further down in the paper suggest that some of the analysis (e.g. the generalised translation of period indices into cohort ones) may be applied to the heterogeneous case as well.

how many births occur to mothers at their young ages. Population economists might focus on the cumulated earnings and savings that people collect between starting their job carriers and marrying. This list may be continued. Generally, knowledge about the distribution of life cycle events among different life stages, *when these stages are relevant to the process under study or to the objectives of research*, may improve our understanding of the demographic process.

When the situation of changing tempo is considered, the quantum within life stages estimated from period data will be distorted compared to the quantum in the birth cohorts (see further down in this paper). In this way we also define *the tempo distortions* as local phenomena and, consequently, will be able to provide a general formula to adjust for these distortions. It was widely discussed in the literature how the adjusted quantum would be related to the cohort quantum. This question is ambiguous when applied to the integral quantum. Yet at the local level, as introduced in this paper, the adjustment may always be interpreted as the translation from the period number of events within a life stage to the respective cohort number of events. (As we build the theory on addressing infinitesimally short life stages, this interpretation does not involve knowledge about the future.)

Importantly, we reveal the purely geometrical nature of distortions to be a local phenomenon. The geometrical theory of tempo distortions is general and does not depend on how exactly the life stages are defined. This may also facilitate the scholarly discussion and agreement on the phenomenon of tempo distortions, as the nature of it is separated here from the definition of what life stages are.

That said, the paper presents some practical approaches to formalising the notion of life stages. The life stages are defined according to the cumulated proportions of demographic events in real birth cohorts (leading to the cohort-oriented study) or in calendar periods (leading to the period-oriented study). Consequently, ages separating the life stages are defined as percentiles of life cycle events. Such a definition provides a generalisation of other approaches proposed in the literature which were formalising the tempo based on summary characteristics of the distribution of life cycle events by age, such as mean age, variance, etc.

We show that the definition of life stages according to the cumulated cohort proportions of life cycle events is equivalent to assuming that the quantum within life stages is subject to cohort-specific proportionate changes (with arbitrary timing of the life stages). Definition of life stages build on the cumulated period proportions of adjusted life cycle events is similarly shown to be equivalent to period-wise proportionate changes in the quantum.

Although the theory presented below is based on defining the main concepts in local terms, we eventually arrive at several important theoretical findings about integral indicators of quantum and of tempo.

In particular, we show that Ryder's translation theory, though originally developed under simplifying assumptions regarding the tempo changes, remains valid under rather general conditions of constant quantum within life stages with arbitrary timing of the life stages. When the quantum is allowed to change in a cohort-specific proportional way, then Ryder's translation formula does not hold but the theory may be generalised and this paper may present the most general translation formula and equation for adjustments of period rates for such a case.

When the period approach is concerned we show that under rather general conditions (namely, when the quantum within life stages is affected to period-specific proportional changes, with arbitrary changes in the tempo) an exact formula for adjusting the observed period integral quantum may be proven which is of the same type as the Bongaarts-Feeney adjustment. An important difference, however, is that it is the dynamics of the mean age of the adjusted age profile of rates, not of the observed mean age, which should be taken into account. Further, we show that it is impossible, using iterations or other data manipulations, to estimate the 'true' adjustment unless an additional assumption is made about how the tempo change occurs or, in better words, about which of the many alternative internally consistent 'true' adjustments we want to obtain. The Bongaarts-Feeney formula, in particular, may be considered as an *exact* adjustment when assuming that tempo changes do not affect the mean age. The Bongaarts Feeney method works well only for the integral indicators of quantum, however. When it comes to estimating age-specific adjustments, the Bongaarts-Feeney method cannot be used, and we provide a method to derive the full age profile of adjusted rates which is consistent with the Bongaarts-Feeney (or any other) adjustment of the integral indictors of the quantum. We also show that results obtained by Kohler and Philipov can be generalised and point to some assumptions and methods necessary to account for higher order moments of age at life cycle event.

Our findings for life stages defined from a cohort and a period perspective imply that both approaches may supplement each other and are not necessarily mutually exclusive. Adjustments based on cumulated cohort proportions of life cycle events provide a *translation* of period indicators into weighted cohort indicators. Adjustments based on the period perspective serve another task; they reveal a dominating period-specific factor of the quantum within life stages in a given period. In both cases, however, the quantum, as defined here, refers to cohorts, i.e. to the number of life cycle events during a particular life stage for the birth cohort passing through the stage at the period of observation.

For traditional purposes of tempo theories, life stages should be defined in a link with actual decisions of people concerning their life cycle events or with other factors (like Bongaarts and Feeney's "life extension pill") envisioned to affect the timing of life cycle events. When people tend to postpone births, for instance, their life stages should accordingly be shifted upwards on the age scale. The theory presented in this paper, however, is also applicable to the consideration of life stages defined in a different context, not necessarily related to life cycle events. One may put, e.g., the analysis of demographic data into an economic context, considering the numbers of life cycle events within life stages to be determined by individual economic developments (starting job, reaching a certain absolute or relative salary level, buying a house, etc.) or—vice versa—one may also put economic analysis in a demographic context, considering ‘numbers’ of economic events (savings, spending, etc.) within demographically determined life stages.

The paper consists of four parts. First, a general geometric interpretation of tempo distortions and a general formula for adjustments are proposed. In Section 2, the theory is supplemented by general formulas for practical calculations, as well as by some useful general formal relations. In Section 3 we demonstrate some general ways to define the life stages based on empirical age distributions of life cycle events and also on hypotheses concerning the quantum changes. In the final section, we illustrate our findings on empirical examples. Some derivations and illustrations are taken to the appendices.

1. TEMPO DISTORTIONS AND ADJUSTMENTS: A GENERAL GEOMETRICAL THEORY AND EXACT FORMULAE

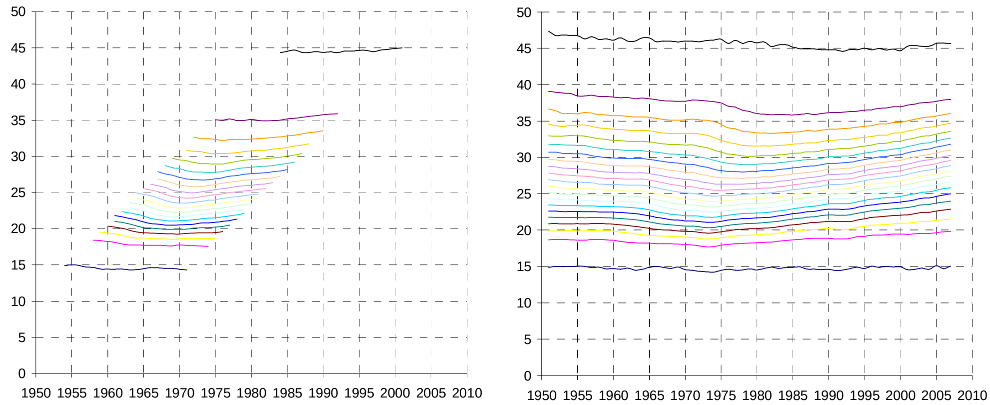
Consider the Lexis diagram, where the horizontal axis represents calendar time and the vertical axis represents age; hence, birth cohorts are represented by diagonal lines originating at time of birth of the cohort.

In addition to the traditional lines representing life cycle events in the age-period space, we represent the tempo of life cycle events by *stage lines*; each stage line showing by what age at each calendar time a birth cohort reaches a certain life stage. The stage lines may form rather arbitrary patterns on the Lexis surface, except that they may not intercept each other. *We will also assume that a stage line may be crossed by cohort or period lines only once, i.e. life stages are unique non-recurrent periods in the history of birth cohorts and also in calendar periods.* This last assumption implies that stage lines may not have slope angles equal to or above 45^{o} (this would result in some life stages being gone through more than once by birth cohorts); nor may they turn backward in time (this would result in some life stages being observed more than once in calendar periods).[2]

Two examples of how the stage lines may look like are presented in Fig. 1 for the case of fertility of the Austrian female population. In these examples, stage lines are approximated by cohort (a) and period (b) *percentiles* of standardised numbers of births (i.e. of fertility incidence rates). The approximation of tempo by percentiles generalises theories proposed by Ryder, Bongaarts and Feeney, who used mean age at childbearing as a tempo indicator and also by Kohler and Philipov, who considered a general approach and, in one particular case, added variance of age at childbearing to the description of the tempo. (We will return to this discussion in the following sections. Note, for the theory presented further down it does not matter how the stage lines are actually defined: from cohort or period perspective or in any other way.)

Figure 1 Cohort and period percentile lines shown on the Lexis surface for fertility rates of the Austrian female population, 1951-2007: 0% to 100% percentiles [a] with step 5%.

a. Cohort percentiles *b. Period percentiles*

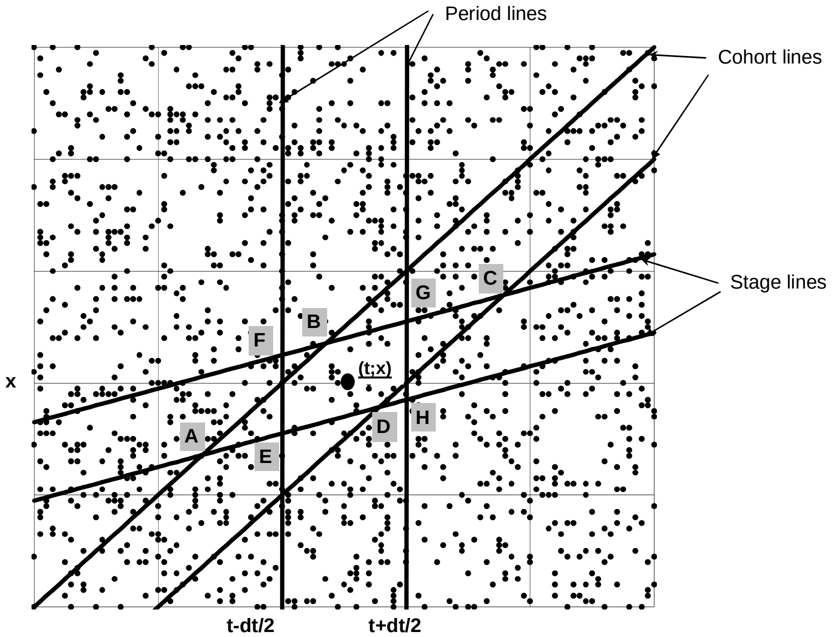


[a] In order to avoid erratic patterns for 0% and 100% percentiles, they are replaced by

[2] The latter restriction may be loosened by working in a transformed Lexis space, where the horizontal axis represents birth cohorts instead of calendar periods.

0.05% and 99.95% percentiles.
Source: Author's computations based on data from the Eurostat (http://epp.eurostat.ec.europa.eu) and Statistics Austria (www.statistik.at) compiled by Tomas Sobotka

Figure 2 Elementary area of the Lexis surface bounded by stage, cohort and time lines

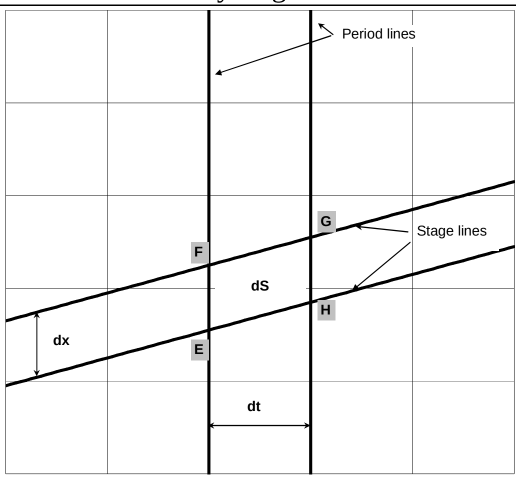


Source: Author's simulations

Let us consider an arbitrarily small vicinity of a point $(t;x)$ on the Lexis surface. Let us consider two stage lines crossing the vicinity of the point (see illustration in Fig. 2; dots on the graph illustrate demographic events under study) and the number of demographic events in cohort and period within the elementary life stage bounded by the stage lines.

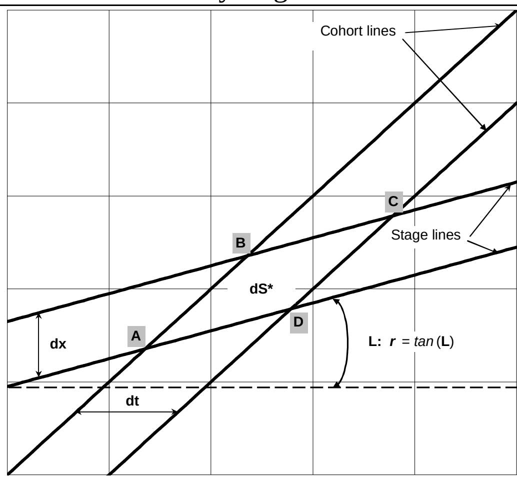
In the arbitrarily small vicinity of interest we may assume stage lines to be parallel to each other, the density of demographic events to be constant, and the number of events happening in subsets of the Lexis surface to be proportional to the area of these subsets. We will characterise the distance between the stage lines by distance *dx* along the age axis separating the stage lines (i.e. by the duration of the elementary life stage from period perspective). Our purpose is to compare the number of life cycle events observed within the elementary life stage during a calendar period of length *dt* with the relevant number of events in a birth cohort born during a similar interval of time *dt*—*x* years prior to the observation (see Fig. 3).

Figure 3a Elementary area of the Lexis surface bounded by stage and time lines



$$dS = dxdt$$

Figure 3b Elementary area of the Lexis surface bounded by stage and cohort lines



$$dS^* = \frac{dxdt}{1-r(x;t)} = \frac{dS}{1-r(x;t)}$$

Source: Author’s simulations

From the period perspective, the number of events occurring during the elementary life stage bounded by two successive stage lines (see Fig. 3a) will be proportional to the area of the quadrangle EFGH which equals the product of the duration of the calendar period (*dt*) and of the period increment of age corresponding to the life stage (*dx*). (See Appendix 1 for derivation.) Hence, from a period perspective, we could expect

$$dE = e(x;t)dxdt \tag{1}$$

events to occur during the life stage determined by the stage lines considered, where $e(x;t)$ is the density of events at the age *x* at calendar time *t*.

Estimates based on observations over cohort, however, are different from this expectation. From a cohort perspective, the number of events during the same life stage (see Fig. 3b) will be proportional to the area of the quadrangle ABCD, which equals the product of the length of the period over which the cohort was born (for the sake of comparability, it must equal the elementary duration chosen for the calendar time *dt*) and of the (same as above) elementary increment of age corresponding to the life stage (*dx*) *divided by one minus tangent of the slope angle of the stage lines*. (See Appendix 1 for the derivation.) Hence, from a cohort perspective, one may expect

$$dE^* = \frac{e(x;t)}{1 - r(x;t)} dxdt \tag{2}$$

events to occur during the life stage considered, where $r(x;t)$ is the tangent of the slope angle (or, equivalently, the derivative over age) of the stage line passing at the age *x* at the calendar time *t*.

Estimates (1) and (2) provide an insight into the nature of tempo-induced distortions in the numbers and intensities of life cycle events: *whenever there is a regular shift of age at which people enter a certain life stage, the number of events of any kind that happen within the life stage will be different depending on whether it was obtained from the history of a real birth cohort or from period observation over adjacent birth cohorts passing through the same life stage in the period under study*.

Another interpretation might be in terms of the 'would-there-be-no-tempo-changes' projection scenario. In such a scenario, the life stages should stop shifting along the age axis, and the stage lines should turn horizontal in the future. If the life stages have a concrete demographic relation to the life cycle events analysed, one may expect that the numbers of events within the life stages of birth cohorts (i.e. of real people) will not alter because of the stop in tempo changes. Hence, while the cohort quantum within each of the life stages will stay at the level determined by (2), period measures of the quantum within life stages will change from the observed level (1) to the cohort level (2). In other words, when the life stages determine the numbers of events happening in real cohorts within the life stages, *observed densities of events will depend on the rates at which the stage lines rise*.

One may also consider an opposite situation where life stages are not linked to the events analysed and the density of events does not change because of the shifts in timing of life stages. In that case, a proper interpretation of the difference between (1) and (2) would be that real birth cohorts and synthetic cohorts from the period observations are exposed to the same events in different ways. (This would apply, for instance, to the study of pension payments: they will look differently from period and cohort perspectives if life expectancy is extending.)

The local distortions of quantum may obviously distort *distribution* of life cycle events among different life stages and across the age scale (distorting, in case of mortality, the indicators of life expectancy) as well as the *integral quantum* over the entire life span (distorting, in case of fertility, indicators of total fertility).

When the shift of life stages is the same at all ages, i.e. the denominator in (2) is age-independent, we may arrive at the integral distortion of the numbers of events over the entire life span, which was considered in classical works by Ryder, Bongaarts and Feeney. The geometrical interpretation presented above, however, shows that the tempo distortion is a phenomenon of local nature and integrates into classical distortions driven by changes in the mean age only in a special case.

The local character of the phenomenon was already pointed out by Kohler and Philipov (2001), who proposed to use the measure of cumulated amount of postponement of births. In case of constant quantum, their theory may be shown to be equivalent to the one presented here. To see that, one may note that the Kohler-Philipov's cumulated amount of postponement by age x at time t may be replaced in our theory by the difference between the percentile passing at point $(t;x)$ and the level of the same percentile in a hypothetical 'no tempo changes' scenario. (A similar concept of increments to life was also proposed by Feeney (2006) to address mortality tempo.) Our theory implies, however, that any reference to such an integral postponement characteristics is unnecessary. Besides, percentiles represent only one way of quantifying the notion of life stages.

The distortion between period and cohort estimates of the number of life cycle events within infinitesimally short life stages may be corrected for by the following adjustment of densities of events, which straightforwardly follows from (1), (2):

$$e^*(x;t) = \frac{e(x;t)}{1-r(x;t)}, \tag{3}$$

where $e^*(x;t)$ is the adjusted density of life cycle events, which—being integrated over a finite life stage or over the entire life span—will bring results consistent with the period summary of relevant numbers of events for birth cohorts observed in the period over the life stage considered.

For an arbitrarily short life stage, the adjustment based on (3) may be interpreted as a *translation* of the period number of events into the cohort number of events during the life stage. Being integrated over a significantly long life stage Ω

$$E^*_\Omega = \int_\Omega e^*(x;t)dx = \int_\Omega \frac{e(x;t)}{1-r(x;t)}dx \tag{4}$$

or over the entire life span, however, the adjusted density should not necessarily coincide with the integral quantum of any of the cohorts passing through the life stage at the period of observation; rather, it is only a synthetic summary of parts of the life spans of birth cohorts which fall into the observation period. Under different conditions, the adjusted integral period quantum may have different interpretations: it may represent translation into cohort quantum; it may indicate the quantum to be observed under the no tempo changes scenario; or it may reflect how favourable the period conditions are for the quantum of cohort life stages which happen to fall within the period. We will study some of the aspects of the integral quantum (4) further down in the paper.

Formally, one may also consider integration of the adjusted density of events (3) over arbitrary subsets on the Lexis surface, not necessarily confined to a particular period of observation. Such an exercise, however, demands more cautious interpretations, as it may imply significant dislocations of life cycle events along the cohort lines as a result of the hypothetical no tempo changes scenario (see discussion and illustration in Appendix 2).

2. PRACTICALITIES: GENERAL CALCULATION FORMULAS AND USEFUL RELATIONS

To derive general relations, it is convenient to quantify the life stages by assigning number s to each of the stage lines. In the examples above, where stage lines are represented by percentiles, each stage line may be associated with the cumulated proportion of life cycle events for the cohort or period. In this case, the youngest stage line will be assigned by zero value and the oldest one will be assigned by unity, with all other stage lines assigned to monotonically increasing values between 0 and 1.

In principle, the choice of values s to quantify life stages is rather arbitrary. It is convenient, however, to assume that these values increase monotonically from the value 0 at birth up to value 1 as the cohort (period schedule) eventually reaches the end of the life course. Assuming monotonicity, we avoid, in particular, two distinct stage lines having the same number assigned to them.[3] (Once this property holds for cohorts, it must also hold for periods and vice versa—due to the assumption above that each stage line may cross period or cohort lines only once.) Indeed, this

[3] This assumption may be loosened, e.g. for life stages, within which no life cycle events of interest are occurring.

assumption still leaves a wide range of alternative options for the choice of the s-function: any monotonic transformation of it, with end values fixed at levels 0 and 1, results in alternative acceptable function of stage lines.

As stage lines may not intercept, numbers s assigned to them form functions of age and of birth cohort/of calendar time; let us denote by $s(x,t)$ and $s(x;t)$ these cohort- and period-wise *stage functions*. We will always separate the calendar time variable by a semicolon from the age variable, while for the cohort's date of birth we will use the comma as a separator, i.e. $s(x,t) = s(x;t+x)$. We assume these functions to be smooth enough, i.e. to have all necessary derivatives.

Let us denote by $x(s,t)$ the age by which the cohort born at time *t* reaches the stage line assigned to the value s. Similarly, by $x(s;t)$ we denote the age at which the stage line nominated by the value s passes the calendar time *t*, i.e. $x(s,t) = x(s;t + x(s,t))$ and $x(s;t) = x(s,t - x(s;t))$. (These functions do exist, as we assumed $s(x,t)$ and $s(x;t)$ to be monotonic functions of age.) Note that $x(s,t)$ and $x(s;t)$ are two distinct functions with different second independent variables. The comma-semicolon notation of cohort- and period-related functions we use here requires some effort to adapt to. Once adopted, however, it provides more clarity and greater conciseness of derivations.

The slope of the stage line passing at age *x* at calendar time *t*, which appears in the adjustments (3), is given by the partial derivative:

$$r(x;t) \overset{def}{=} r(x,t-x) = \frac{\partial x(s(x;t);t)}{\partial t}. \tag{5}$$

To obtain a more practical formula for this derivative and also for other relations, the following derivation may be used (be reminded that expressions with comma and semicolon represent different functions):

$$\frac{\partial x(s,t)}{\partial s} = \frac{dx(s;t+x(s,t))}{ds} = \frac{\partial x(s;t+x(s,t))}{\partial s} + \frac{\partial x(s;t+x(s,t))}{\partial t}\frac{\partial x(s,t)}{\partial s} =$$

$$= \frac{\partial x(s;t+x(s,t))}{\partial s} + r(x(s,t);t+x(s,t))\frac{\partial x(s,t)}{\partial s}, \qquad (6)$$

i.e. the following useful links between derivatives may be established:

$$\frac{\partial x(s,t)}{\partial s} = \frac{1}{1-r(x(s,t),t)}\frac{\partial x(s;t+x(s,t))}{\partial s}, \qquad (7)$$

$$\frac{\partial x(s;t)}{\partial s} = (1-r(x(s;t);t))\frac{\partial x(s,t-x(s;t))}{\partial s}, \qquad (8)$$

$$\frac{\partial s(x,t)}{\partial x} = (1-r(x,t))\frac{\partial s(x;t+x)}{\partial x}, \qquad (9)$$

$$\frac{\partial s(x;t)}{\partial x} = \frac{1}{1-r(x;t)}\frac{\partial s(x,t-x)}{\partial x}, \qquad (10)$$

and the following expression for the adjusted rates may be obtained from (3):

$$e^*(x;t) = e(x;t)\frac{s'_x(x;t)}{s'_x(x,t-x)}. \qquad (11)$$

Hence, for instance, the exact general expression for the period integral quantum adjusted for tempo distortions is given by:

$$TE^*(t) = \int_0^\infty e(x,t-x)\frac{s'_x(x;t)}{s'_x(x,t-x)}dx. \qquad (12)$$

(This expression for the integral quantum assumes that it is a sum of individual rates; in the case of life table computations, individual adjusted rates (11) may similarly be used to obtain any of the adjusted table functions.)

For practical purposes, relation (11) may be approximated by the finite differences:

$$e^*(x;t) \approx e(x;t)\frac{s\left(x+\frac{\Delta}{2};t\right)-s\left(x-\frac{\Delta}{2};t\right)}{s\left(x+\frac{\Delta}{2};t+\frac{\Delta}{2}\right)-s\left(x-\frac{\Delta}{2};t-\frac{\Delta}{2}\right)}, \tag{13}$$

here Δ is the interval, over which the average derivative is approximated. In fertility tempo studies, for instance, erratic variations may usually be prevented by taking three-year differences around period/age of interest, i.e. by setting $\Delta = 3$ in (13).[4]

The adjusted period integral quantum may be approximated as

$$TE^*(t) \approx \sum_x E(x;t)\frac{s\left(x+\frac{\Delta}{2};t\right)-s\left(x-\frac{\Delta}{2};t\right)}{s\left(x+\frac{\Delta}{2};t+\frac{\Delta}{2}\right)-s\left(x-\frac{\Delta}{2};t-\frac{\Delta}{2}\right)}, \tag{14}$$

here $E(x;t)$ are observed age-specific rates at age group *x*.

Similarly, general relations may be written for other adjusted period indicators, such as mean age at event (life expectancy at birth, as it would be called for life table calculations) etc.

These equations are of direct practical use. When, for example, stage lines are represented by percentiles, i.e. for a generalised version of the usual assumption that tempo is defined by the proportions of life cycle events at different ages, one may write:

$$E^*(x;t) = \frac{E(x;t)}{1-r(x;t)} = E(x;t)\frac{p'_x(x;t)}{p'_x(x,t-x)} \approx$$

$$\approx E(x;t)\frac{p\left(x+\frac{\Delta}{2};t\right)-p\left(x-\frac{\Delta}{2};t\right)}{p\left(x+\frac{\Delta}{2};t+\frac{\Delta}{2}\right)-p\left(x-\frac{\Delta}{2};t-\frac{\Delta}{2}\right)}, \tag{15}$$

where $p(x;t)$ are cumulated proportions of standardised life cycle events by age *x* at time *t*, obtained from cohort schedules (which would generalise Ryder's approach) or from calendar schedules (which would generalise Bongaarts and Feeney's approach; see, however, a more detailed discussion below). Consequently, the general formula for the adjusted period integral quantum would be:

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[4] Bongaarts and Feeney (2000) have also recommended a similar triennial approximation for a derivative in their formula.

$$TE^*(t) \approx \sum_x E(x;t) \frac{p\left(x+\frac{\Delta}{2};t\right) - p\left(x-\frac{\Delta}{2};t\right)}{p\left(x+\frac{\Delta}{2};t+\frac{\Delta}{2}\right) - p\left(x-\frac{\Delta}{2};t-\frac{\Delta}{2}\right)}. \qquad (16)$$

These formulas are *general* and were derived without any assumptions regarding the dynamics of the age profile of the demographic rates, unlike in other works where this profile was assumed either to shift (works by Ryder, Bongaarts and Feeney), to shift and change its variance (works by Kohler and Philipov as well as Kohler and Ortega) or to follow a specific model (Yi and Land 2001).

For studying the distribution of the quantum between life stages and also for formulating assumptions about the processes of tempo changes, it is convenient to introduce the following *quantum function* which reflects the notion of local quantum as the number of events in birth cohorts per unit change of the stage function:

$$q(s,t) = \frac{d}{ds} \int_0^{x(s,t)} e(y,t)dy = e(x(s,t),t) \cdot x'_s(s,t). \qquad (17)$$

For the sake of clarity, one may also write the quantum function as a function of age:

$$q(x,t) \stackrel{def}{=} q(s(x,t),t) = \frac{e(x,t)}{s'_x(x,t)} = \frac{e^*(x,t)}{s'_x(x;t+x)}. \qquad (18)$$

In period notations, we may write equivalently:

$$q(x;t) \stackrel{def}{=} q(s(x;t);t) = q(x,t-x) = \frac{e(x;t)}{s'_x(x,t-x)} = \frac{e^*(x;t)}{s'_x(x;t)}. \qquad (19)$$

The adjusted period integral quantum may be derived from (19) as follows:

$$TE^*(t) = \int_0^\infty e^*(x;t)dx = \int_0^\infty q(x;t)s'_x(x;t)dx = \int_0^1 q(s;t)ds, \qquad (20)$$

i.e. the adjusted period integral quantum is a synthetic summary of local quantum of the birth cohorts observed in the calendar period (note that nonnegative weights $s'_x(x;t)$ integrate to unity: $\int_0^\infty s'_x(x;t)dx = \int_0^1 ds = 1$).

Note that we do not introduce the function of local quantum of calendar periods. Being introduced as $q^P(s;t) = \frac{d}{ds}\int_0^{x(s;t)} e(y;t)dy = e(x(s;t);t)\cdot x'_s(s;t)$, it would be distorted compared to the quantum of the birth cohorts:

$$q^P(x;t) = q(x;t)\cdot\frac{x_s'(s(x;t);t)}{x_s'(s(x;t),t-x)} = q(x;t)\cdot\frac{s_x'(x;t)}{s_x'(x,t-x)}. \qquad (21)$$

Such a line of reasoning could provide another approach to derive distortions: compare (21) to (11).

3. Several particular cases of practical interest where the stage lines may be inferred based on distributions of life cycle events. Relations between period and cohort integral quantum

In this section we present several practical ways to define life stages and establish general relations which provide some insight into the links between cohort and period integral quantum. The results presented also shed a new light on the underlying assumptions and the applicability of the traditional methods for adjustment proposed by Ryder, Bongaarts and Feeney and others. First, we discuss the aforementioned approximations, when tempo is defined according to the cumulated proportions of life cycle events. Then we consider an approach where the model is assumed about how the quantum within life stages is affected to cohort- or period-specific factors. Interestingly, in both approaches it is somehow possible to show that life stages may be defined according to the cumulated proportions of life cycle events.

Before proceeding with particular definitions of life stages, it is worth reminding that the theory and relations presented above are general and independent of any considerations which might have been put in the definition of the life stages. A study of the same process may be put into different contexts, which may be characterised by different definitions of life stages. As the research agenda may vary considerably, definitions of life stages may not be limited to those presented below, even though they seem to be of significant importance.

For the sake of simplicity, we assume further down the densities of the observed events to be *standardised* according to the size of the cohort. (For fertility it may be done based on incidence rates; for mortality, cohorts' life table densities of deaths may be used.) The theory may in fact be applied to the crude numbers as well; yet, totals of crude numbers would make little demographic sense.

Life stages defined according to the cumulated proportions of life cycle events

We start with the aforementioned examples where the tempo of life cycle is directly linked to the distribution of life cycle events by age. Consequently, life stages are defined according to the cumulated proportions of life cycle events—either in cohorts or in periods. Adjustments in this case are given by Eqs. (15) and (16), which are both general and practical.

The theories proposed in the literature so far have also been based on associating tempo changes with changes in the characteristics of the distributions of life cycle events by age. Mean age at birth, at death or, generally, at life cycle event are the classical characteristics of age distribution used in the literature. Under shifting hypothesis or otherwise assuming distortions being age-independent, it is clear that the general formulas (15), (16) will converge to the traditional translation and adjustment equations (as the slope of percentiles will coincide with the derivative of any characteristics of the position of the age distribution along the age scale). The picture of tempo changes may, however, be more complicated. In particular, changes in the variance may also play an important role as was demonstrated by Kohler and Philipov (2001), see also Kohler and Ortega (2002). In this case, as well as in other cases of non-classical tempo changes, Eqs. (15), (16) will again capture the effects relevant for distortions of the quantum at each individual age.

It is interesting to clarify what the adjustments implied by defining the life stages according to cumulated proportions would mean substantively. A more clear case is that of the life stages defined according to the cohort cumulated proportions. Suppose we study mortality. In that case cohort cumulated proportions of deaths would be equal to numbers dying in the normalised cohort by a particular age. When we consider a life stage of the birth cohort during which the cumulated proportion of deaths increases from, say level A to the level B, we know that B-A gives the number of deaths in the respective period in a normalised cohort. Consider, however, the same life stage from the period perspective. If we consider the number of deaths (normalised to the size of the respective cohort) in the calendar period in between ages, when the respective cohorts' cumulated proportions of deaths equal A and B, this number may deviate from B-A, as those two levels of the cumulated proportion will be observed in two different cohorts. If there are tempo changes, these cohorts may differ with respect to the cumulated proportions of deaths. Adjustments (3) allow to correct the period numbers of (normalised) deaths in such a way that they become consistent with cohort proportions: the adjusted number of events observed in the age group corresponding to the change in the cohorts' cumulated proportion of deaths from level A to level B will exactly equal B-A. Considering all life stages, starting from age 0, it is clear that by any age the adjusted number of (normalised) deaths will always cumulate to the same number it has actually cumulated to in the life course of a real birth cohort, which reaches the same age in the period of observation. As the cumulated proportion of deaths supplements to one the survival probability, one may note that the tempo-adjusted numbers of deaths normalised to the size of birth cohorts will constitute a 'life table' composed of survival probabilities of real birth cohorts observed in the period $l_p^*(x;t) \equiv l_c(x, t-x)$. A life table composed of such survivorship probabilities would produce life expectancy equal to what is known as CAL, cross-sectional average length of life (e.g. Brouard 1986; Guillot 2003, 2006). Note, however, that in period life table computations we are not interested in the *numbers* of deaths as such. Rather, we use them to compute age specific incidence rates, from which we construct a synthetic life table. Hence, the aforementioned relation between survival probabilities does not, in fact, define tempo-adjusted period life table; it only illustrates one aspect of the adjustments implied by defining the life stages according to the cumulated cohort proportions of deaths. For a general type of life cycle events, a similar correspondence applies to the cumulated proportions of events in the overall cohort quantum: after adjustment, proportions of events in the overall quantum of the respective cohort cumulated over calendar period for up to an age x will match to the cumulated proportion of the cohort, which reaches the age x in the period of observation.

When period proportions are used to define the life stages, interpretation of the relevant adjustments is less straightforward. They may be considered as *indirect* adjustments serving a similar purpose, as those based on cohort proportions. To see that, consider, first, adjusting cohort numbers, rather than period ones, in order to reconcile the correspondence between cohort and period observations (see (3)):[5]

$$e(x;t) = e^{*}(x;t) \cdot (1 - r(x;t)). \qquad (22)$$

When the adjustments are done according to the period cumulated proportions of events, proportions of events in the overall quantum of respective periods cumulated along the cohort line will match corresponding proportion cumulated over the calendar period. Original adjustments (3), then, are indirect adjustments of period numbers using the same adjustment

[5] Such adjustment could make sense if densities of events are not linked to the life stages and stay at their observed values in the no-tempo-changes scenario. In such a case, cohort numbers, instead of the period numbers, of events within life stages will change under new exposure times implied by the no-tempo-changes scenario.

factor, which would have been used to adjust the cohort numbers in order to reconcile them with period proportions.

Interestingly, the adjustments done according to the cumulated cohort or period proportions also appear in a different context where life stages are defined based on assumptions about factors driving the quantum, as presented below.

Let us consider an approach where instead of explicitly defining the life stages we assume a model about the dynamics of the quantum within life stages, i.e. about factors driving the numbers of life cycle events within life stages of birth cohorts. First we address the simplest case, where the quantum within life stages is constant, and continue by addressing the situation, where the quantum is driven by cohort-specific or period-specific factors.

Case of constant quantum within life stages

When the locally defined quantum is constant, i.e.

$$q(s;t) = q(s,t) \equiv q(s) \tag{23}$$

and all changes in life-course distribution of events are due to changes in the tempo, there is direct correspondence between the adjusted period integral quantum and the cohort integral quantum (which naturally will be constant as well as the local quantum):

$$TE^{*}(t) = \int_{0}^{1} q(s;t)ds \equiv \int_{0}^{1} q(s)ds \equiv CE \,. \tag{24}$$

In other words, the adjusted period integral quantum may be interpreted as the translation of period quantum into cohort quantum.

This correspondence may be extended to any part of the life cycle, within which the quantum is not altered:

$$TE_{[s1;s2]}{}^{*}(t) = \int_{s1}^{s2} q(s;t)ds \equiv \int_{s1}^{s2} q(s)ds \equiv CE_{[s1;s2]}, \tag{25}$$

here $TE_{[s1;s2]}{}^{*}(t)$ and $CE_{[s1;s2]}$ are the adjusted period quantum and cohort quantum for the life stages, where the stage function changes from the value *s1* to the value *s2*.

Other important findings for the case of constant quantum within life stages are that (i) stage function may equivalently be replaced by cumulated cohort proportions of life cycle events, i.e. relations (15)-(17) may be used; (ii) similarly, cumulated period proportions of *adjusted* densities of events may also be used (in fact, they will coincide with cohort proportions); (iii) adjustments for the integral quantum may be obtained in a simpler way, following Ryders' translation or BF's adjustment formulas, which are based, however, on mean age at life cycle event of the *adjusted* age profile of densities of life cycle events. (These results follow from more general results shown further down for the cases of cohort-driven and period-driven proportional changes in the quantum.)

Although assumption (23) looks rather simplistic, it may be justified for the case where cohort integral quantum is constant, for instance, in case of mortality, where the integral quantum is one for each of the birth cohorts. In such a case, it is always possible to formally assume that quantum of all the life stages is constant, i.e. any redistributions of life cycle events are due to tempo changes. Even though such an assumption may look counterintuitive in some cases, results (24), (25) imply that adjustments obtained under such an assumption will provide the exact translation of period totals into cohort ones. Exact general formulas for these translations are given by (15), (16) based on cumulated cohort proportions of events.

Cohort-specific proportional changes in the quantum: the generalised Ryderian case

A more general assumption about quantum change would be to assume that varying conditions affect the quantum proportionally at all life stages. We consider first an intuitively and mathematically clearer case of cohort-driven proportional changes in the quantum, which leads to a general version of Ryder's approach. Further down, the case of period-driven changes will also be studied.

Let us assume that the quantum is subject to cohort-specific proportional changes only, i.e.

$$q(s,t) = CE(t) \cdot \alpha(s), \tag{26}$$

where, by the scaling agreement, $\int_0^1 \alpha(s)ds = 1$. (One may check that the cohort integral quantum may be correctly obtained from (26): $CE(t) = \int_0^1 q(s,t)ds = CE(t)\int_0^1 \alpha(s)ds = CE(t)$.)

Under assumption (26), the adjusted period integral quantum may be obtained as the weighted average of the birth cohorts' integral quantum:

$$TE^*(t) = \int_0^\infty q(x;t)s'_x(x;t)dx = \int_0^\infty CE(t-x)\alpha(s(x;t))s'_x(x;t)dx =$$

$$= \int_0^\infty CE(t-x)w(x;t)dx, \tag{27}$$

where non-negative weights $w(x;t) = \alpha(s(x;t))s'_x(x;t)$ sum up to unity: $\int_0^\infty w(x;t)dx = \int_0^\infty \alpha(s(x;t))s'_x(x;t)dx = \int_0^1 \alpha(s)ds = 1$.

Hence, assuming only cohort-like proportional changes in the quantum, one may arrive at Ryder's interpretation of adjusted period integral quantum as an approximation of the cohort integral quantum, averaged over cohorts observed in the period. The exact formulation of this interpretation is provided by (27).

Another general result for the cohort-driven case of quantum is that the stage function may be equivalently replaced by cumulated proportions of the events observed in the birth cohort. Indeed, any monotonic transformation of the stage function also results in a stage function, which defines the same stage lines. Therefore, to prove the aforementioned result, one may show that the cohort proportions of life cycle events form a monotonic transformation of the stage function where (26) holds:

$$p^c(x,t)=\frac{\int_0^x e(y,t)dy}{\int_0^\infty e(y,t)dy}=\frac{\int_0^{s(x,t)} q(u,t)du}{\int_0^1 q(u,t)du}=\frac{CE(t)\int_0^{s(x,t)} \alpha(u)du}{CE(t)\int_0^1 \alpha(u)du}=$$

$$=\int_0^{s(x,t)} \alpha(u)du = g(s(x,t)). \qquad (28)$$

Note the non-negativity of the slope of the transformation: $g'(s)=\alpha(s)\geq 0$, it may equal zero only for age groups in which no life cycle events are observed. Therefore, cohort proportions $p^c(x,t)=g(s(x,t))$ may be used to define the same life stages as reflected by $s(x,t)$, except for stages with zero quantum. Since stages with zero quantum have no effect on the adjusted quantum, using cohort proportions (28) will result in the same adjustments as implied by the stage function $s(x,t)$.

Interestingly, the reverse result is also valid: the definition of the life stages according to the cumulated cohort proportions of events leads to (26). Indeed, $s(x,t)=p^c(x,t)$ implies that $s'_x(x,t)=e(x,t)/CE(t)$ (see (28)); and from this result and from Eq. (18) it follows that $q(s;t)\equiv CE(t)$, i.e. $\alpha(s)\equiv 1$.

Hence, the definition of life stages according to the cumulated cohort proportions of life cycle events is possible if and only if the quantum is assumed to be subject to cohort-specific proportional changes as described by (26). Remember that this assumption does not mean that the age distribution of cohorts' life cycle events must have a constant shape, as the assumption (26) is formulated in terms of life stages, and not in terms of the physical age, i.e. it allows for arbitrary changes in the timing of life stages.

This result supports the aforementioned simple way to define the life stages according to the cumulated proportions of life cycle events in birth cohorts. It also implies that (16) based on cumulated cohort proportions is the most general adjustment formula for the Ryderian case of arbitrary tempo changes under cohort-driven proportional changes in the quantum.

Another implication, which follows from (27) is that adjustments based on life stages defined according to the cumulated cohort proportions provide the exact translation of period totals into (weighted) cohort totals. Hence, Eq. (16) is possibly the most general translation equation when cohort proportions are used in the formula.

Interestingly, a formula similar to Ryder's translation equation for converting period integral quantum into cohort quantum and vice versa, based on changes of the mean age at life cycle event, may be developed in a rather general case of tempo changes. This result, however, may be obtained only under a fixed integral quantum, i.e. in fact, in the aforementioned case of a constant local quantum (this is a particular consequence of the results presented further down for the period-driven quantum case). By contrast, in a period-driven scenario of quantum changes a more general result may be established. This implies that the Bongaarts-Feeney theory for adjusting the period integral quantum based on changes of the period mean age at life cycle event may be considered to have a wider applicability. We will return to this discussion further down.

In some cases fully proportional cohort changes in quantum may be considered unfeasible. The quantum of mortality, for instance, at child and old ages may respond differently to changes in life conditions. (Feeney (2006) has raised this issue in the concluding part of his paper by suggesting mixed models to address the mortality tempo.) The quantum of fertility at young childbearing ages, in the middle of the childbearing or at 'late-fertility' ages may also respond differently to a changing environment. Therefore, a more general case might be of practical importance, where the quantum is assumed to change proportionally within parts of birth cohorts' life stages only. Let us assume that the proportionality of quantum change applies to a limited part of the life cycle:

$$q(s,t) = CE_{[s1;s2]}(t) \cdot \alpha(s), \; s \in [s1;s2], \tag{29}$$

where $\int_{s1}^{s2} \alpha(s)ds = 1$ and $CE_{[s1;s2]}(t) = \int_{s1}^{s2} q(s,t)ds$ is the cohort integral quantum for the life stages covering $s \in [s1;s2]$. Under assumption (29), the adjusted period integral quantum for stages $s \in [s1;s2]$ may be obtained as the weighted average of the relevant part of cohort quantum:

$$TE^*{}_{[s1;s2]}(t) = \int_{x(s1;t)}^{x(s2;t)} CE_{[s1;s2]}(t-x)\alpha(s(x;t))s'_x(x;t)dx =$$

$$= \int_0^{\infty} CE_{[s1;s2]}(t-x)w(x;t)dx, \tag{30}$$

where non-negative weights $w(x;t) = \alpha(s(x;t))s'_x(x;t)$ sum up to unity: $\int_{x(s1;t)}^{x(s2;t)} w(x;t)dx = \int_{x(s1;t)}^{x(s2;t)} \alpha(s(x;t))s'_x(x;t)dx = \int_{s1}^{s2} \alpha(s)ds = 1$. Similar to the complete cohort-wise proportionality case, a stage function within values $s \in [s1;s2]$ may also be equivalently replaced by a function based on the cumulated proportions of life cycle events:

$$p_{[s1;s2]}{}^c(x,t) = s1 + (s2 - s1) \cdot \frac{\int_{x(s1,t)}^{x} e(y,t)dy}{\int_{x(s1,t)}^{x(s2,t)} e(y,t)dy} = s1 + (s2 - s1) \cdot \frac{\int_{s1}^{s(x,t)} q(u,t)du}{\int_{s1}^{s2} q(u,t)du} =$$

$$= \frac{CE_{[s1;s2]}(t) \int_{s1}^{s(x,t)} \alpha(u)du}{CE_{[s1;s2]}(t) \int_{s1}^{s2} \alpha(u)du} = \int_{s1}^{s(x,t)} \alpha(u)du = g_{[s1;s2]}(s(x,t)) \cdot \tag{31}$$

Hence, it is possible to extend the general translation relations to the case of considering parts of the life cycle based on cumulated cohort proportions of life cycle events.

Period-specific proportional changes in the quantum

Now, let us consider period-wise proportional changes in the quantum within life stages:

$$q(s;t) = Q(t)\cdot \alpha(s), \tag{32}$$

where, by the scaling agreement, $\int_0^1 \alpha(s)ds = 1$. (This assumption was already employed by Kohler and Philipov 2001, Eq. (2).)

Function $Q(t)$ in (32) is in fact identical to the adjusted period integral quantum:

$$TE^*(t) = Q(t)\int_0^1 \alpha(s)ds = Q(t). \tag{33}$$

Hence, in the period-proportionality case (32) the integral adjusted period quantum may be interpreted as an estimate of the period-specific factor of the quantum; in this sense, it may be considered as reflecting the *period conditions*.

In the case of period-wise proportionality of quantum changes—contrary to the cohort case considered above—the cohort quantum is a weighted average of the adjusted period quantum:

$$CE(t) = \int_0^\infty e(x,t)dx = \int_0^\infty q(x,t)s'_x(x,t)dx =$$

$$= \int_0^\infty Q(t+x)\alpha(s(x,t))s'_x(x,t)dx = \int_0^\infty Q(t+x)w(x,t)dx\,, \tag{34}$$

where non-negative weights $w(x,t) = \alpha(s(x,t))s'_x(x,t)$ sum up to unity: $\int_0^\infty w(x,t)dx = \int_0^\infty \alpha(s(x,t))s'_x(x,t)dx = 1$.[6] Bongaarts and Feeney (2006) showed this relation for the case where tempo distortions do not depend on age and the constant shape assumption holds.

[6] Further down, however, we will show that this relation might have only a limited practical value.

Similar to the cohort-driven case, assumption (32) implies that the stage function $s(x;t)$ may be replaced by the cumulated period proportions of (adjusted) life cycle events

$$p^{p*}(x;t) = \frac{\int_0^x e^*(y;t)dy}{\int_0^\infty e^*(y;t)dy} = \frac{\int_0^{s(x;t)} q(c;t)dc}{\int_0^1 q(c;t)dc} = \frac{Q(t)\int_0^{s(x;t)} \alpha(c)dc}{Q(t)\int_0^1 \alpha(c)dc} =$$

$$= \int_0^{s(x;t)} \alpha(s)ds = g(s(x;t)). \qquad (35)$$

Reversely, using adjusted period proportions (35) to define life stages, automatically results in period-proportionate changes in the quantum (32). Indeed, $s(x,t) = p^{p*}(x;t)$ implies that $s'_x(x;t) = \frac{e^*(x,t)}{TE^*(t)}$ (see (35)); and from this result and from Eq. (19) it follows that $q(s;t) \equiv TE^*(t)$, i.e. $\alpha(s) \equiv 1$.

Hence, the definition of life stages according to the cumulated period proportions of *adjusted* *densities of life cycle events is possible if and only if the quantum is assumed to be subject to period-specific proportional changes as described by (32).*

This result supports the aforementioned simple way to define the life stages according to the cumulated proportions of life-time events in the calendar period. However, it also implies that these are *the adjusted numbers, not the observed ones*, which should be used in defining the life stages as well as in adjustment formulas. An approach based on distribution of the observed rates may only be considered to be an approximation to the result based on distribution of adjusted rates. (Note that these distributions will be identical in the case where the classical assumption of age-independent distortions is valid, see (35).)

Another implication is that Eq. (16) based on cumulated proportions of adjusted period observations is the most general adjustment formula for the integral quantum in the case of arbitrary tempo changes under period-driven proportional changes in the quantum. The necessity to define the life stages which are to be used in making the adjustments, based on the already adjusted rates, complicates the usage of (16). In fact, as we argue further down, it is impossible to resolve this problem without introducing an additional assumption; importantly and, in a sense, unexpectedly, we will see that there is always room for one and only one additional assumption. The Bongaarts-Feeney method, for example, may be supported when assuming that the mean age of the adjusted profile of period rates changes at the same rate as the observed mean age (note, this does not imply shifting hypotheses or any other assumptions concerning the period distribution of life cycle events, except for mean age). To see this, we need the results which are established in the next paragraphs.

When cumulated proportions (35) are used as the stage function, $s(x;t) = p^{p*}(x;t)$, one may derive the following relation for the observed (unadjusted) period integral quantum (note, $e^*(x;t) = s'_x(x;t) \cdot TE^*(t)$; $s'_x(x;t) = s'_x(x,t-x) - s'_t(x,t-x) = s'_x(x,t-x) - s'_t(x;t)$):

$$TE(t) = \int_0^\infty e(x;t)dx = \int_0^\infty e^*(x;t)\frac{s'_x(x,t-x)}{s'_x(x;t)}dx = \int_0^\infty TE^*(t)s'_x(x,t-x)dx =$$

$$= TE^*(t)\int_0^\infty (s'_x(x;t) + s'_t(x;t))dx = TE^*(t)\int_0^\infty s'_x(x;t)dx + TE^*(t)\int_0^\infty s'_t(x;t)dx =$$

$$= TE^*(t) + TE^*(t)\frac{d}{dt}\int_0^X s(x;t)dx = TE^*(t) - TE^*(t)\frac{d}{dt}\mu^*(t), \qquad (36)$$

where X is the upper age limit, after which no life cycle events occur (i.e. $s'_t(x;t) \equiv 0$, $x \geq X$, which allows differentiating by time outside the integral in (36)) and $\mu^*(t) = X - \int_0^X s(x;t)dx = \int_0^X x \cdot s'_x(x;t)dx = \int_0^X x \cdot \frac{e^*(x;t)}{TE^*(t)}dx$ is the mean age at life course event according to *adjusted period densities*. Hence, we have the following analogue to either the Bongaarts-Feeney adjustment or to Ryder's translation:

$$TE^{*}(t)=\frac{TE(t)}{1-\frac{d}{dt}\mu^{*}(t)}. \tag{37}$$

Note that the only assumption underlying (37) is the period-driven proportional change in the quantum within life stages. We do not assume either a fixed age shape of rates or linearity of postponement or any other limiting assumptions with regard to the tempo changes. A result similar to (37) was shown by Kohler and Philipov (2001) for a special case where tempo changes are restricted to a specific assumption (Eq. (10) in the cited work) in addition to assuming the period-specific proportional changes in the quantum. It was also shown by Guillot (2003) for the case of mortality, where the adjusted mean age is understood as CAL, implying that the quantum within separate life stages is constant. Here we have shown that (37) is a direct consequence of the assumption of period-wise proportional changes in the quantum, without any additional assumptions made.

Hence, it is possible to derive a general equation relating an observed period integral quantum to the adjusted period integral quantum based only on the information on the mean age at life cycle event obtained from the period schedule. However, it must be the already *adjusted* schedule, not the observed one. Indeed, some iterative procedures could be thought of in order to make practical use of (16) and (35) or of (37). Such iterative approaches to improve the Bongaarts-Feeney adjustments were already proposed, among others by Kohler and Philipov (2001), albeit in a different context. Our next results, however, show that it is impossible to point out 'the' one adjustment consistent with the data and with the period-proportionality of quantum assumption. In fact, there are an indefinite number of different adjustments of this type, and the choice among them requires an additional assumption to be made. Second, our results will show that the Bongaarts-Feeney adjustment may be considered an *exact* adjustment under a specific assumption concerning the mean age at life cycle event, which is internally consistent and is valid under much looser assumptions than those the method is usually based on.

In order to see that there are many alternative adjustments consistent with the data and period-proportionality of quantum changes, one may note that for any assumed level of the adjusted period integral quantum there exists a corresponding adjustment. Let $TE^*(t)$ be an *arbitrary* function reflecting the dynamics of the adjusted period integral quantum. Assuming period-proportionality assumption (32) and, therefore, using cumulated proportions (35) as the stage function $s(x;t)$, we have the relation (again, recall $e^*(x;t) = s'_x(x;t) \cdot TE^*(t)$, which follows from (35)):

$$e(x;t) = e^*(x;t)\frac{s'_x(x,t-x)}{s'_x(x;t)} = TE^*(t)s'_x(x,t-x), \qquad (38)$$

i.e. $s'_x(x,t) = \dfrac{e(x;t+x)}{TE^*(t+x)}$ and

$$s(x,t) = \int_0^x s'_x(y,t)dy = \int_0^x \frac{e(y;t+y)}{TE^*(t+y)}dy, \qquad (39)$$

which explicitly determines the stage function and, therefore, the adjustment corresponding to the arbitrarily chosen dynamics of the adjusted period integral quantum[7].

Formally, Eq. (39) implies an additional constraint to be imposed on the adjusted period integral quantum, since the stage function should reach the value of 1 as age increases, i.e.

$$\int_0^\infty \frac{e(y;t+y)}{TE^*(t+y)}dy \equiv 1. \qquad (40)$$

When age-specific intensities of events do not change over time, i.e. $e(x;t) \equiv e(x;t_0)$, this equation may be solved, for instance, at $TE^*(t) \equiv TE(t)$.

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[7] In practical calculations, Eq. (39) is to supplemented by correction procedure at advanced ages in order to assure monotonicity of the stage function. Such correction prevents adjusted rates from turning negative and usually does not distort significantly the adjusted integral quantum.

In general, however, solving (40) presents a tough problem. It is possible to show that Eq. (40) may be transformed into Volterra's integral equation of the first kind which, however, happens to be incorrect and requires regularity constraints to be imposed or additional a priori information to be introduced about the solution (see Appendix 3). From a practical point of view the problem may be reflected as follows. Although an arbitrary choice for the adjusted integral quantum in (39) may violate the constraints $s(x,t) \leq 1$, $s(\infty,t) = 1$, such violations will usually take place at advanced ages, where intensities of events are low and the impact of data errors, of rounding errors, of violations of the model assumptions, of effects of smoothing, etc., are high. Therefore, such violations (i) may not be considered significant; (ii) may easily be offset by corrections to the profile of the stage function $s(x,t)$, which may distort the actual dynamics of the adjusted quantum $TE^*(t)$ from the one assumed in (39) by only a minor margin. Practical calculations show that Eqs. (39), (40)—even under fixed age-specific rates $e(x;t) \equiv e(x;t_0)$—allow for a nearly arbitrary choice of the desired level of the adjusted integral quantum. In the case of fertility, our calculations based on realistic profiles of fertility rates demonstrate the possibility of voluntarily varying the level of the adjusted TF from less than 1% of the 'real' value to nearly as high as 100 times more than that 'real' value. (See details in Appendix 3.)

Summarising, we note that the assumption about the adjusted integral quantum may not be judged from (40). In fact, this assumption may be introduced rather arbitrarily without significantly violating the equation. This additional freedom of choice differentiates the period approach from the cohort approach to model the proportionately changing quantum, as in the latter case the adjustment is fully determined by the proportionality assumption, as discussed above. Apart from many implications, which we discuss further down, this finding also implies that the relation between period and cohort integral quantum (34) may, practically, be rather non-obliging, and of formal value only, as it is closely linked to (40).[8,9]

Unlike Eq. (40), however, Eq. (39), is of practical importance, as it explicitly allows to derive the age-specific distortions and adjustments from the assumed level of the adjusted period integral quantum. Once the stage function is determined from the dynamics of the adjusted integral quantum and from (39), one may explicitly derive the adjusted rates:

$$e^*(x;t) = s'_x(x;t) \cdot TE^*(t) = TE^*(t) \cdot \frac{d}{dx}\left[\int_0^x \frac{e(y;t-x+y)}{TE^*(t-x+y)}dy\right]. \qquad (41)$$

Hence, *the period-proportionality assumption—unlike the cohort-proportionality assumption—of the quantum changes is not sufficient to uniquely define the adjusted profile of life cycle events*. As the value of the adjusted period integral quantum $TE^*(t)$ may be chosen arbitrarily and is sufficient to define all the age-specific adjustments, one would need <u>*one*</u> additional assumption or constraint in order to define the adjustment.

The Bongaarts-Feeney adjustment may present an important case where the extra assumption needed to define the adjustment is *the equality of change rates of the adjusted and of the observed mean ages at life cycle events*. In this case, one may substitute the observed mean age instead of the adjusted mean age into the general equation (37) and define the adjusted integral quantum using a simple formula proposed by Bongaarts and Feeney. In this sense, the Bongaarts-Feeney adjustment is <u>exact</u> and does not need the shifting assumption or any other assumptions concerning the tempo changes except for those related to the observed and adjusted mean ages.

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[8] Weights in (34) will be very sensitive to any kind of errors, smoothing and rounding and to deviations from the proportionality assumption, especially at advanced ages with low intensities of events.

[9] There will be no such freedom of choice, of course, if life stages are pre-defined from independent considerations, e.g. from cumulated (unadjusted) period proportions of events; in such case, however, the period-proportionality assumption (32) may fail to fit the data.

That said, our results imply that the Bongaarts-Feeney adjustment may only be applied to the *integral* period quantum, as supported by (37). It may not be used to characterise the quantum of separate life stages or to derive age-specific adjustments of the observed rates whenever the shifting hypothesis is not consistent with the data. To estimate the adjusted quantum of separate age groups or life stages, one may use Eq. (41). The exercise of estimating the age-specific adjusted rates may be of considerable importance. In the context of demographic policies oriented toward young couples or in the context of housing policies, it could be important not to estimate the Total Fertility but its subtotals for different age categories. Population replacement characteristics, such as the net reproduction rate and Lotka's *r* also involve distribution of fertility by age. In other applications, it might be not the overall total of the quantum as such, but its distribution by age, which is of importance. This definitely is the case for mortality studies where life expectancy measures, which are usually of interest, reflect the distribution of the mortality rates, derived from numbers of deaths, by age. The ability to extend the adjustment method by calculating age-specific adjustments may also help in guessing whether the underlying hypothesis of period-wise proportional changes in the quantum fits our knowledge about the underlying processes of postponement and advancement. (See the following section for some illustrations of how the adjusted age profile may differ from that implied by the shifting hypothesis.)

Kohler and Philipov (2001) proposed their method as an improvement to the Bongaarts-Feeney method by including the effect of variance in addition to the shift in the fertility timing. As follows from the results presented above, the Bongaarts-Feeney method is in fact not necessarily linked to the shifting hypothesis and presents one of many internally consistent adjustments which fully reflect the data. This includes the case where data show arbitrary changes in the variance. (However, we will see that the Bongaarts-Feeney method implies absence of the effects of the *adjusted* variance.) Hence, we must give an interpretation to the Kohler-Philipov method within a broad picture we have come to. As our results imply the possibility of choosing nearly arbitrary values for the adjusted integral quantum, the Kohler-Philipov method (which is also about adjusting the integral period quantum, although its foundation was developed from age-specific distortions) may also be thought of as one of many alternative adjustments. The necessary additional assumption, which must be made in order to specify the adjustment consistent with the period-wise proportionality assumption, is expressed in the Kohler-Philipov method in terms of cumulated tempo (Eq. (10) in the authors' paper). That assumption may be considered rather unrealistic, though, especially at the ends of the fertile period. It implies, for instance, that the upper age limit for fertility increases even faster than the mean age at childbearing, which is not consistent with the data.

Interestingly, though, just like in the case of the Bongaarts-Feeney method, an interpretation to results by Kohler and Philipov can be provided without using their original assumption about cumulated tempo. Before showing that, we first obtain some additional results for the case of period-driven proportional changes in the quantum. These results shed additional light on the Bongaarts-Feeney and Kohler-Philipov methods and also provide a general approach to considering the effects of moments of high orders.

Let us consider the adjusted mean age (note usage of (11) and (38)):

$$\mu^*(t)=\int_0^\infty x\frac{e^*(x;t)}{TE^*(t)}dx=\int_0^\infty x\frac{e(x;t)}{TE^*(t)}\frac{s'_x(x;t)}{s'_x(x,t-x)}dx=\int_0^\infty x\cdot s'_x(x;t)dx \qquad (42)$$

and the observed unadjusted mean age (note usage of (37) and (38)):

$$\mu(t)=\int_0^\infty x\frac{e(x;t)}{TE(t)}dx=\frac{TE^*(t)}{TE(t)}\int_0^\infty x\frac{e(x;t)}{TE^*(t)}dx=\frac{TE^*(t)}{TE(t)}\int_0^\infty x\cdot s'_x(x,t-x)dx=$$

$$=\frac{\int_0^\infty x\cdot\left(s'_x(x;t)+s'_t(x;t)\right)dx}{1-\frac{d\mu^*(t)}{dt}}=\frac{\int_0^\infty x\cdot s'_x(x;t)dx+\int_0^\infty x\cdot s'_t(x;t)dx}{1-\frac{d\mu^*(t)}{dt}}=$$

$$= \frac{\mu^*(t) + \frac{d}{dt}\int_0^X x \cdot s(x;t)dx}{1 - \frac{d\mu^*(t)}{dt}}. \tag{43}$$

The last integral in (43) may be obtained from the following recurrent relation:

$$J = \int_0^X x \cdot s(x;t)dx = x^2 \cdot s(x;t)\Big|_0^X - \int_0^X x \cdot \frac{d}{dx}(x \cdot s(x;t))dx =$$

$$= X^2 - \int_0^X \left(x \cdot s(x;t) + x^2 \cdot s'_x(x;t)\right)dx = X^2 - \int_0^X x \cdot s(x;t)dx -$$

$$- \int_0^X x^2 \cdot s'_x(x;t)dx = X^2 - J - \int_0^X x^2 \cdot s'_x(x;t)dx, \tag{44}$$

i.e.

$$J = \frac{1}{2}\left(X^2 - \int_0^X x^2 \cdot s'_x(x;t)dx\right) = \frac{1}{2}\left(X^2 - \sigma^{*2}(t) - \mu^{*2}(t)\right), \tag{45}$$

where

$$\sigma^{*2}(t) = \int_0^X \left(x - \mu^*(t)\right)^2 \cdot s'_x(x;t)dx = \int_0^X x^2 \cdot s'_x(x;t)dx - \mu^{*2}(t) \tag{46}$$

is the variance of the adjusted age profile of events' densities.

Substituting (45) into (43), we have:

$$\mu(t) = \frac{\mu^*(t) - \frac{1}{2}\frac{d}{dt}\left(\sigma^{*2}(t) + \mu^{*2}(t)\right)}{1 - \frac{d\mu^*(t)}{dt}} = \frac{\mu^*(t) - \mu^*(t)\frac{d\mu^*(t)}{dt} - \frac{1}{2}\frac{d\sigma^{*2}(t)}{dt}}{1 - \frac{d\mu^*(t)}{dt}} =$$

$$= \mu^*(t) - \frac{\frac{d}{dt}\sigma^{*2}(t)}{1 - \frac{d}{dt}\mu^*(t)}. \tag{47}$$

This is a general result which shows explicitly the role of the adjusted variance.

The result established by Kohler and Philipov (2001, Result 8, Eq. (13) for linear changes and Result 10 for non-linear changes) is a particular case of (47) where the dynamics of the mean age and of the variance is of special type (Eq. (10), Result 6 from the aforementioned paper).

Interestingly, however, the general relation (47) must also be satisfied by the Bongaarts-Feeney adjustment. The solution for this paradox lies in the fact that the Bongaarts-Feeney formula, when applied to data with arbitrary dynamics of the variance and supplemented by Eq. (41) to derive the age-specific adjustments, will null the effects of the *adjusted* variance, which would be another reflection of the internal consistency of the method.

As Kohler and Philipov pointed out, in some occasions it may seem more realistic to assume significant changes in the adjusted variance. In that case, (47) could be used as a general result. Indeed, one still needs to define the adjusted variance in (47). One choice would be to replace the adjusted variance by the observed one and solve (47) for the adjusted mean age. As it is the derivative of the mean age, which is necessary for the adjustment calculations, one may use the following relations:

$$\tfrac{d}{dt}\mu^*(t) = \tfrac{d}{dt}\mu(t) + \frac{\tfrac{d^2}{dt^2}\sigma^{*2}(t)}{1-\tfrac{d}{dt}\mu^*(t)} + \frac{\tfrac{d}{dt}\sigma^{*2}(t)}{\left(1-\tfrac{d}{dt}\mu^*(t)\right)^2}\tfrac{d^2}{dt^2}\mu^*(t), \qquad (48)$$

or, in terms of the adjustment coefficient $K \stackrel{def}{=} \dfrac{1}{1-\tfrac{d}{dt}\mu^*(t)}$,

$$1 - K\left(1-\tfrac{d}{dt}\mu(t)\right) + K^2\tfrac{d^2}{dt^2}\sigma^{*2}(t) + K^3\tfrac{d}{dt}\sigma^{*2}(t)\cdot\tfrac{d^2}{dt^2}\mu^*(t) = 0\,. \qquad (49)$$

If (a) the adjusted variance is constant, or (b) both the adjusted variance and the adjusted mean age are linear functions of time, (48), (49) will anyway converge to the Bongaarts-Feeney adjustment.

Approximating the second derivative of the mean age as well as derivatives of the variance by the observed values, one may find the adjustment coefficient as a solution of the following cubic equation:

$$1 - K\left(1-\tfrac{d}{dt}\mu(t)\right) + K^2\tfrac{d^2}{dt^2}\sigma^{2}(t) + K^3\tfrac{d}{dt}\sigma^{2}(t)\cdot\tfrac{d^2}{dt^2}\mu(t) = 0\,. \qquad (50)$$

(In practice, the last summand is usually of minor value, compared to the others and, therefore, a simpler quadratic approximation may be used:

$$1 - K\left(1-\tfrac{d}{dt}\mu(t)\right) + K^2\tfrac{d^2}{dt^2}\sigma^{2}(t) + {K_{BF}}^3\tfrac{d}{dt}\sigma^{2}(t)\cdot\tfrac{d^2}{dt^2}\mu(t) \approx 0;\ K_{BF} \stackrel{def}{=} \frac{1}{1-\tfrac{d}{dt}\mu(t)}.)$$

One could also use higher order differentiation, if approximations in (50) are not feasible. In any case, we will obtain one of many alternative internally consistent adjustments, which are not improving the accuracy of each other; rather, they are based on different additional assumptions which therefore are better being done explicitly.

The technique proposed above regarding the incorporation of the effect of the variance may also be used to follow the effects of moments of higher order $m_k^*(t) = \int_0^X \left(x - \mu^*(t)\right)^k \cdot s'_x(x;t)dx$ and $m_k(t) = \int_0^X \left(x - \mu(t)\right)^k \cdot s'_x(x, t-x)dx$ using derivations similar to (42)-(47). Considering the discrepancy between observed and adjusted moments of order *k* reveals effect of the moment of order *k*+1.

The Bongaarts-Feeney, Kohler-Philipov and other adjustments mentioned above are only several of many possibilities consistent with the data and period-wise proportionality assumption. One may develop other adjustments based, for instance, on assuming the adjustment with the least possible distortions of the observed rates (minimising, e.g., the sum of squares of deviations between observed and adjusted rates), with the least possible distortions of the cumulated proportions of the observed rates, with the least possible tempo changes, etc. We will not follow up this issue here, although we point out its potential importance. Instead, we proceed now to the case where period-wise proportionality may be assumed for subsets of the life cycle only.

When period-wise proportionality (32) may be assumed only for a part $s \in [s1; s2]$ of the life course, an analogue of (34) may be derived for the appropriate life stages. Also, the stage function may be replaced by an equivalent based on the cumulated proportions of the adjusted period numbers of life cycle events (here both the starting and end values of the stage function and the integral quantum must correspond to that part of the life course which the analysed age belongs to, i.e. $x \in [x(s_1;t), x(s_2;t)]$):

$$s(x;t) = p_{[s1;s2]}{}^{p*}(x;t) = s1 + (s2 - s1) \cdot \frac{\int_{x(s1,t)}^{x} e^{*}(y;t)dy}{\int_{x(s1,t)}^{x(s2,t)} e^{*}(y;t)dy} . \qquad (51)$$

It is also possible to extend the relations between integral quantum and adjusted mean age, using derivations similar to (36):

$$TE_{[s1;s2]}(t) = TE^{*}{}_{[s1;s2]}(t)\left(1 - \frac{d}{dt}\mu^{*}_{[s1;s2]}(t)\right), \qquad (52)$$

where $TE_{[s1;s2]}(t) = \int_{x(s1;t)}^{x(s2;t)} e(x;t)dx$, $TE^{*}{}_{[s1;s2]}(t) = \int_{x(s1;t)}^{x(s2;t)} e^{*}(x;t)dx$ and $\mu^{*}_{[s1;s2]}(t) = \int_{x(s1;t)}^{x(s2;t)} x \cdot \frac{e^{*}(x;t)}{TE^{*}_{[s1;s2]}(t)} dx$ is the mean age at life cycle event within the life stage corresponding to $s \in [s1; s2]$. That is, the adjustment for the integral quantum of the sub-set of life course may be obtained based on the mean age at life cycle event within the sub-set:

$$TE^{*}{}_{[s1;s2]}(t) = \frac{TE_{[s1;s2]}(t)}{1 - \frac{d}{dt}\mu^{*}_{[s1;s2]}(t)} . \qquad (53)$$

Other results obtained earlier may also be extended to the case of period-wise proportionality of quantum changes within the subset of life cycle:

$$s(x,t) = s_1 + \int_{x(s1;t)}^{x} \frac{e(y;t+y)}{TE^{*}{}_{s(x;t+y)}(t+y)/(s_2 - s_1)} dy . \qquad (54)$$

$$e^{*}(x;t) = s'_x(x;t) \cdot \frac{TE^{*}{}_{s(x;t)}(t)}{s_2 - s_1} = TE^{*}{}_{s(x;t)}(t) \frac{d}{dx} \int_{x(s1;t)}^{x} \frac{e(y;t+y)}{TE^{*}{}_{s(x;t+y)}(t+y)} dy . \qquad (55)$$

Notes on applying the proportionate-quantum change assumptions to the rates of the first kind (events-to-exposure)

As it follows from the analysis presented in the subsections above, the life stages defined according to the cumulated proportions of life cycle events are intrinsically linked to assumptions about cohort- or period-wise proportional changes in the quantum within life stages. Such an assumption may appear rather unrealistic for the case where the dynamics of life cycle events are eventually modelled based on events-to-exposure rates. To see that, note that our proportionality assumptions (26) and (32) relate the number of events during the life stage to cohort- or period-specific factor (i.e. to cohort or period ‘conditions’) and to the life stage itself. In a framework of events-to-exposure rates, however, the number of events is also related to the population exposed.

A proper proportionality assumption for the case of events determined by both the cohort/period factor, by life stage and by the exposure, would be

$$q(s,t) = M(t)l(x(s,t),t)\alpha(s) \text{ or } q(x,t) = M(t)l(x,t)\alpha(s(x,t)) \quad (56)$$

for the cohort-driven dynamics case (here $M(t)$ is the cohort factor of the quantum, and $l(x,t)$ is the exposure, for instance, population survived, by age *x* in cohort *t*) and

$$q(s;t) = M(t)l(x(s;t),t - x(s;t))\alpha(s) \text{ or } q(x;t) = M(t)l(x,t - x)\alpha(s(x;t)) \quad (57)$$

for the period-driven dynamics case (here $M(t)$ is the period factor of the quantum; note the usage of exposures from real cohorts).

As a scaling agreement, we will assume that

$$\int_0^1 \alpha(s)ds = 1 \quad (58)$$

in both cases.

Events-to-exposure rates may be written as (note usage of (19))

$$\varepsilon(x;t) = \varepsilon(x,t - x) = \frac{e(x;t)}{l(x,t - x)} = \frac{q(x;t)s'_x(x,t - x)}{l(x,t - x)} \quad (59)$$

(e.g. in the case of mortality this would correspond to the force of mortality).

For the cohort case (56) this leads to

$$\varepsilon(x,t) = \frac{q(x,t)s'_x(x,t)}{l(x,t)} = M(t)\alpha(s(x,t))s'_x(x,t). \quad (60)$$

Integrating (60) and normalising by $\int_0^X \varepsilon(y,t)dy = M(t)\int_0^1 \alpha(u)du = M(t)$

(recall, X is the upper limit of the life span, after which no life cycle events may occur or they may be neglected[10]) we obtain a monotonic transformation of the stage function:

$$\frac{I(x,t)}{M(t)} = \frac{\int_0^x \varepsilon(y,t)dy}{M(t)} = \int_0^x \alpha(s(y,t))s'_x(y,t)dy = \int_0^{s(x,t)} \alpha(u)du\,, \tag{61}$$

i.e. the stage function may equivalently be replaced by

$$s(x,t) = \frac{\int_0^x \varepsilon(y,t)dy}{M(t)}\,. \tag{62}$$

Similarly, in the case (57) of period-driven changes we have:

$$\varepsilon(x;t) = \frac{q(x;t)s'_x(x,t-x)}{l(x,t-x)} = M(t)\alpha(s(x;t))s'_x(x,t-x)\,, \tag{63}$$

and, since events-to-exposure rates will be subject to the same adjustment coefficients as the events' numbers (exposure at the moment of observation is not altered by tempo changes at the moment itself):

$$\varepsilon^*(x;t) = \varepsilon(x;t)\frac{s'_x(x;t)}{s'_x(x,t-x)} = M(t)\alpha(s(x;t))s'_x(x;t)\,, \tag{64}$$

and the monotonic transformation

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[10] Note that in some cases, as in that of mortality, there is no formal upper limit to the life span. Extending integration of the force of mortality to an infinite life span will indeed create an obstacle to the formalisation we present. However, one may overcome this problem by setting a rather high formal limit X to the life span and neglecting any events afterwards. This may be interpreted as assuming that after age X there are no tempo changes, i.e. the stage function is constant as a function of time at age X and above. Since, there is no need for adjusting at these ages, we may focus the analysis on ages below X only and set the stage function at age X at the value of one. In the case of mortality such an interpretation is supported by the fact that at oldest old ages mortality is rather stable. Note also that shifting the upper age X does not alter the structure of stage lines, as it only scales up or down the stage function by a constant coefficient. Also, note the possibility of considering a shifting upper limit X(t), which we are not doing here, however.

$$\frac{J(x;t)}{M(t)} = \frac{\int_0^x \varepsilon^*(y;t)dy}{M(t)} = \int_0^x \alpha(s(y;t))s'_x(y;t)dy = \int_0^{s(x;t)} \alpha(u)du\,, \qquad (65)$$

indicates that the stage function may equivalently be defined as

$$s(x;t) = \frac{\int_0^x \varepsilon^*(y;t)dy}{\int_0^X \varepsilon^*(y;t)dy} \qquad (66)$$

(note $M(t) = \int_0^X \varepsilon^*(y;t)dy$ due to the scaling agreement).

Hence, in the case where numbers of life cycle events are assumed to depend on the cohort/period factor, on the life stage and also on the population exposed, the life stages must be defined according to the cumulated cohort/period proportions of events-to-exposure rates. It is also evident that in this case the general relations established above for the integral quantum will hold for the integrals of the events-to-exposure rates and not for the integrals of the densities of life cycle events. In particular the translation relations may be applied to integrals of the events-to-exposure rates. Similarly, the discussion presented above about a variety of adjustments in the period-driven case as well as about justifications for the usage of the Bongaarts-Feeney formula, etc., should also be applied to integrals and to mean ages of the age schedules of event-to-exposure rates, not to the numbers of events normalised by the size of the birth cohort. In the case of mortality, for instance, this means that one should use the mean age according to the schedule of the force of mortality and not of the mean age at death.

The last, but not least, approach to defining life stages relevant for a process described by event-to-exposure rates might be based on the observation that different age groups (hence, possibly, different life stages) respond in different ways to changing cohort or period conditions (e.g. Gómez de León 1990; Lee and Carter 1992; Ediev 2008). Thus mortality at oldest old ages was resistant to mortality reduction in many countries. In such a case, the quantum may be assumed to follow the models:

$$q(s,t) = (\alpha(s) + k(t)\beta(s))l(x(s,t),t) \tag{67}$$

in cohort-driven case and

$$q(s;t) = (\alpha(s) + k(t)\beta(s))l(x(s;t),t - x(s;t)) \tag{68}$$

in period-driven case (note the resemblance of the model structure proposed in the aforementioned works by Gómez de León, Lee and Carter) with scaling agreements:

$$\int_0^1 \alpha(u)du = 0\,,\ \int_0^1 \beta(u)du = 1\,. \tag{69}$$

Following the same techniques as above, one may show that the stage function may be defined under (67), (68) as

$$s(x,t) = \frac{\int_0^x \varepsilon(y,t)dy - \int_0^x \varepsilon(y,t_0)dy}{\int_0^X \varepsilon(y,t)dy - \int_0^X \varepsilon(y,t_0)dy} \tag{70}$$

in the cohort-driven case and as

$$s(x;t) = \frac{\int_0^x \varepsilon^*(y;t)dy - \int_0^x \varepsilon^*(y;t_0)dy}{\int_0^X \varepsilon^*(y;t)dy - \int_0^X \varepsilon^*(y;t_0)dy} \tag{71}$$

in the period-driven case, t_0 referring here to some base cohort or period (in case of the period-driven quantum one may consider a period with supposedly no tempo changes as such a base period and, consequently, substitute the adjusted base rates by observed base rates in (71)).

So far it is hard to judge which proportionality assumption—involving the exposure or not—is more relevant to considering the dynamics of life cycle events which are commonly described in terms of the events-to-exposure rates. As our analysis indicates, using assumptions without exposures involved will help reconciling period and cohort (or adjusted period) numbers of events as such. The appealing feature of this kind of adjustment is that it ensures that the period total of deaths normalised to the size of the birth cohort being 'translated' into a cohort number is always one. Making an assumption about quantum changes involving the exposure, on the other hand, seems to be intuitively more appealing, although it reconciles only the period and cohort totals of the events-to-exposure rates. These totals may lack significant demographic meaning. In the case of mortality, for instance, the integral of the force of mortality over the life span has no important demographic meaning and, therefore, any 'translation' converting period levels of this integral into cohort levels looks rather unnatural compared to the alternative translation of the number of deaths during the life course.

No matter what approach is preferred, the analysis presented in the paper provides the tools to obtain the necessary adjustments. The approach of deriving the life stages from an assumption about the factors affecting the quantum may also be developed in other contexts. The central element of the approach is finding a monotonic transformation between the unobserved stage function and observed quantities. In our case, these were cumulated proportions of events or of rates, which formed monotonic functions of the value of the stage function; other quantities may serve the same purpose in other contexts.

4. EMPIRICAL ILLUSTRATIONS

In this section we demonstrate the practical use of the results presented above. We do not aim at thoroughly studying any of the empirical examples presented. Rather, our purpose is only to illustrate the theoretical findings presented here by means of empirical examples.

The section is subdivided into two parts. In the first subsection we address applications for adjusting observed mortality rates and in the second subsection we present examples for the case of fertility. We start with mortality because based on the theoretical results presented above it happens to be easier to address due the known cohort integral quantum (it equals one as everybody dies and only once).

As indicated above, the theory may be applied to any kind of life cycle events (not necessarily, in fact, of demographic nature). Hence, the examples based on fertility and mortality data below represent only some of many possible fields of applications.

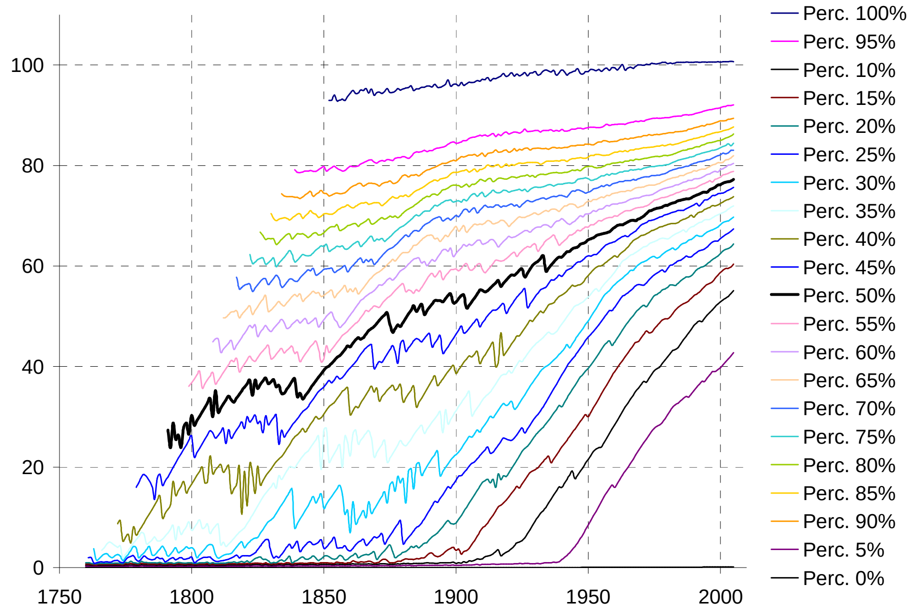
We limit our illustrations to those cases where the quantum is assumed to follow either cohort-wise or period-wise proportional changes. That is, we limit ourselves to the generalised versions of Ryder's cohort approach and Bongaarts-Feeney's period approach. This limitation is rather artificial, however, as other researchers may define the life stages and stage lines on a different basis (e.g. from sociological or medico-biological considerations, from a population economics point of view, etc.), in which case the general findings presented above (Eqs. (1)-(22)) may be used.

Mortality

In the case of mortality, the cohort integral quantum is known, since the number of deaths over the entire life cycle of a birth cohort standardised by its size at birth, equals one. Therefore it is rather simple to apply the adjustments (15) based on cumulated cohort proportions of deaths, as these proportions may be calculated even before the cohort completes its life course. Percentiles for cohort distributions of deaths obtained from cohort life tables for Sweden's male population covering the calendar periods 1759 to 2005 are presented in Fig. 4. (These and other data on mortality were downloaded from the *Human Mortality Database*[11] in period format and roughly converted into cohort schedules by rearranging quadrangles on the Lexis surface.)

[11] The *Human Mortality Database*, sponsored by University of California, Berkeley (USA) and Max Planck Institute for Demographic Research (Germany), may be accessed at www.mortality.org or www.humanmortality.de.

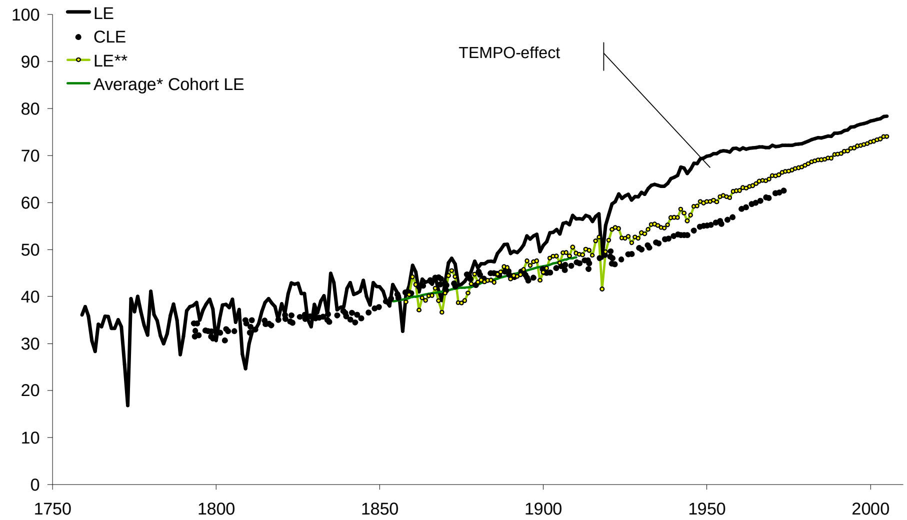
Figure 4 Percentiles of deaths age distributions for birth cohorts of Swedish males, calendar periods 1759-2005



Source: Author's calculation based on data from the *Human Mortality Database* (www.mortality.org or www.humanmortality.de)

As seen from Fig. 4, the tempo of deaths was changing quite rapidly in past centuries, albeit with significant fluctuations. It is also clear that the dynamics of percentiles do not support shifting or any other simplified hypothesis about changes in age distributions of deaths—neither for the entire age span, nor for its subsets. Therefore we must apply the general formula based on age-specific adjustments (15).

Figure 5 Observed period (LE) and cohort (CE), adjusted period (LE^{**}) and averaged cohort life expectancies at birth for Sweden, males, 1759-2005. Adjustments are based on cumulated cohort proportions of deaths. Adjustments are based on estimating derivatives by 7-year differences.



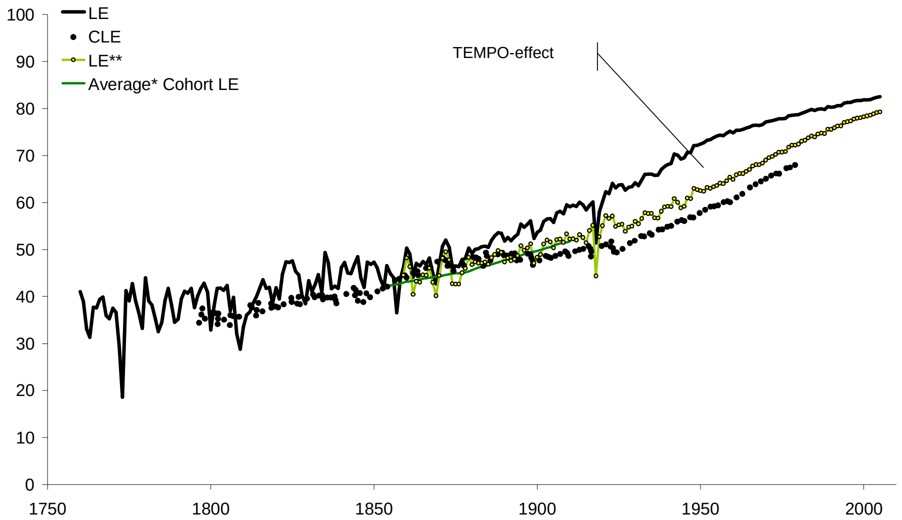
Source: Author's calculation based on data from the *Human Mortality Database* (www.mortality.org or www.humanmortality.de)

Life expectancies at birth (LE^{**}) obtained from period life tables based on adjusted age-specific death rates are shown in Fig. 5. Adjustments were obtained according to (15) based on cohort cumulated proportions with derivatives approximated by 7-year differences.[12] The unadjusted period life expectancies at birth (LE) as well as the cohort life expectancies (CLE) are shown in the same figure (the latter indicator being shifted along the time axis by a period equal to the cohort's life expectancy). Also, the figure depicts weighted averages (27) of cohorts' life expectancies with weights corresponding to the proportions of the adjusted numbers of deaths in the

[12] For data from the historical past, when mortality was high and volatile, smoothing by means of the 7-years long frame helps to avoid downward biases of life expectancy due to stochastic fluctuations; for modern data, even single-year approximations to derivatives provide the same results.

integral quantum of the relevant birth cohorts. (Weighting simply by period proportions of deaths normalised by the size of the cohort provides a very close result.) In principle, the translation relation (27) is to be applied only to the adjusted integral quantum (i.e. to the Total Mortality Rate, as it is called by Bongaarts and Feeney (2006)) and not to the life expectancy at birth. Still, a good correspondence between the adjusted life expectancy and the weighted average for cohort life expectancies suggests that the adjusted life expectancy is also a good approximate for the weighted average of life expectancies of birth cohorts observed in the calendar period. Interestingly, the adjusted life expectancy may be computed even for recent calendar periods, although it would be impossible to directly obtain weighted averages of cohort indicators until the cohorts complete their life courses. Therefore, adjustments to mortality rates may provide a useful insight into cohorts' weighted life expectancy, even though not all cohorts observed have completed their life course yet. The difference between observed and adjusted life expectancies indicates the tempo effect on life expectancy at birth. If the life stages of cohorts defined according to the cumulated proportions of deaths stopped shifting along the age scale, the observed period life expectancy would drop to the adjusted level. At the same time, the adjusted period life expectancy remains to be an indicator of the period, as it is a composite reflection of mortality of the birth cohorts observed in the period; the adjusted age pattern of period mortality, generally speaking, may not be observed in any of the real birth cohorts.

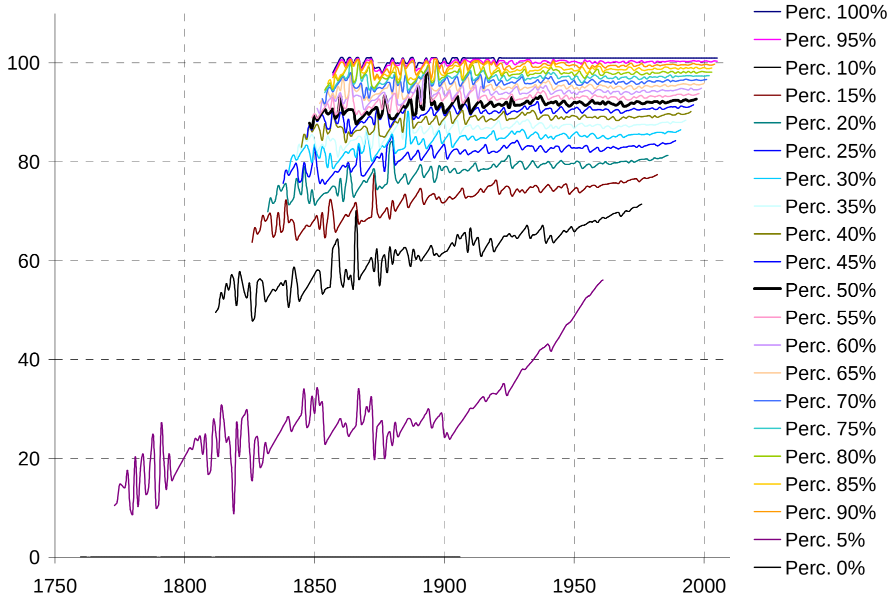
Figure 6 Observed period (LE) and cohort (CE), adjusted period (LE^{**}) and averaged cohort life expectancies at birth for Sweden, females, 1759-2005. Adjustments are based on cumulated cohort proportions of deaths. Adjustments are based on estimating derivatives by 7-year differences.



Source: Author's calculation based on data from the *Human Mortality Database* (www.mortality.org or www.humanmortality.de)

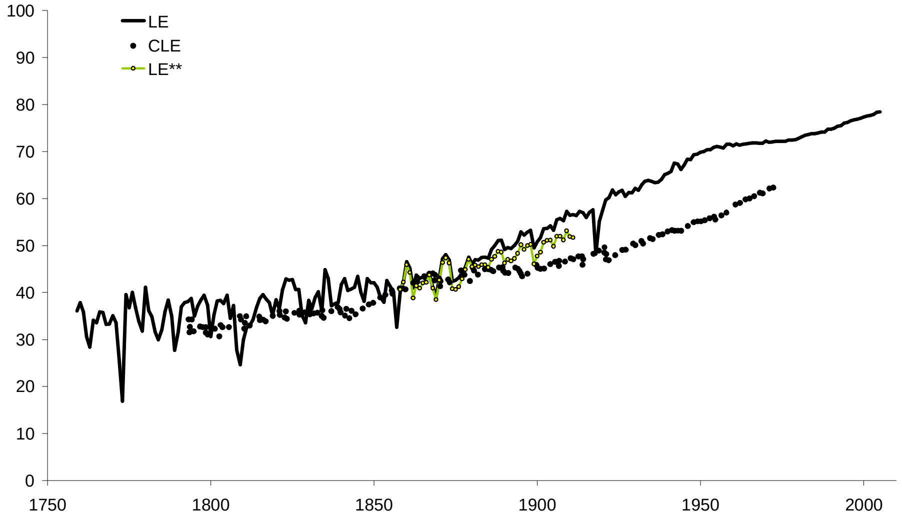
In Fig. 6 we present similar results for Swedish female population. In Figs. 7 and 8 we present patterns of the life stages and adjustments based on an exposure-specific assumption (56) about the quantum. We note less relevance of such a translation for the purpose of guessing about cohorts' life expectancy and its lower ability to deal with incomplete cohorts. It also to be noted that age profiles of the adjusted rates based on exposure-specific translations tend to be rather erratic, unless some smoothing procedures are applied (we recommend, for instance, to smooth the stage function before taking its derivatives). Yet a more thorough comparison of the different approaches to defining the life stages presented in the third part of the paper deserves additional studies. (Note that we do not present here any example for the case of mortality, which would be based on a period-specific quantum changes hypothesis.)

Figure 7 Percentiles based on cumulated integrals of death rates for birth cohorts of Swedish males, calendar periods 1759-2005



Source: Author's calculation based on data from the *Human Mortality Database* (www.mortality.org or www.humanmortality.de)

Figure 8 Observed period (LE) and cohort (CE) and adjusted period (LE**) life expectancies at birth for Sweden, males, 1759-2005. Adjustments are based on cumulated cohort proportions of deaths. Adjustments are based on estimating derivatives by 7-year differences.



Source: Author's calculation based on data from the *Human Mortality Database* (www.mortality.org or www.humanmortality.de)

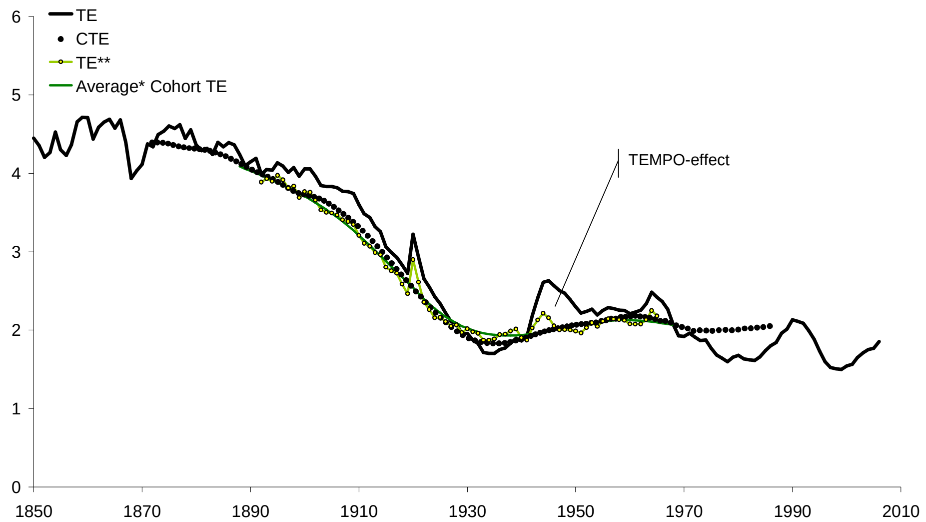
Fertility

Adjustments of the period TFR based on cumulated cohort proportions of fertility are illustrated in Fig. 9 (we use period age-specific fertility rates for Sweden,[13] 1850-2006, rearranging them in order to obtain cohort profiles). As one may see from the illustration, the adjustment does indeed provide a good way of translation (some deviations from the weighted cohort total are only due to approximation errors of derivatives). Such a translation, however, is of limited practical importance as it requires the observed cohorts to complete their fertility history before the translation may be carried out. (Other than the case of mortality where the method could

[13] Data are taken from Festy (1979) for the period 1850-1960 and from the Eurostat web page (http://epp.eurostat.ec.europa.eu) for the rest of the period.

be applied to all periods, as the cohort integral quantum was known equal unity.) However, one may overcome this shortcoming by (i) guessing about possible levels or ranges for the cohort TFR; (ii) extrapolating the stage lines for incomplete cohorts, as these lines usually show a regular pattern; or (iii) studying cohorts' life stages from sociological, cultural, economic and other considerations.

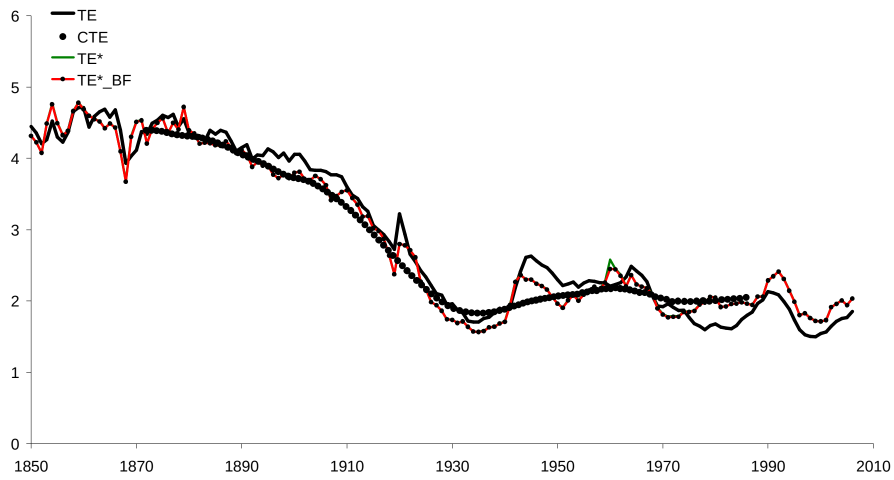
Figure 9 Observed period (TE) and cohort (CTE), adjusted according to the cumulated cohort proportions (TE^{**}) and averaged cohort total fertility rates for Sweden, 1850-2006. Adjustments are based on estimating derivatives by 3-year differences. Observed cohort rates are shifted by a period equal to the cohort mean age at childbearing.



Source: Author's calculation based on data from Festy (1979) for the period 1850-1960 and from the Eurostat web page (http://epp.eurostat.ec.europa.eu) for the rest of the period

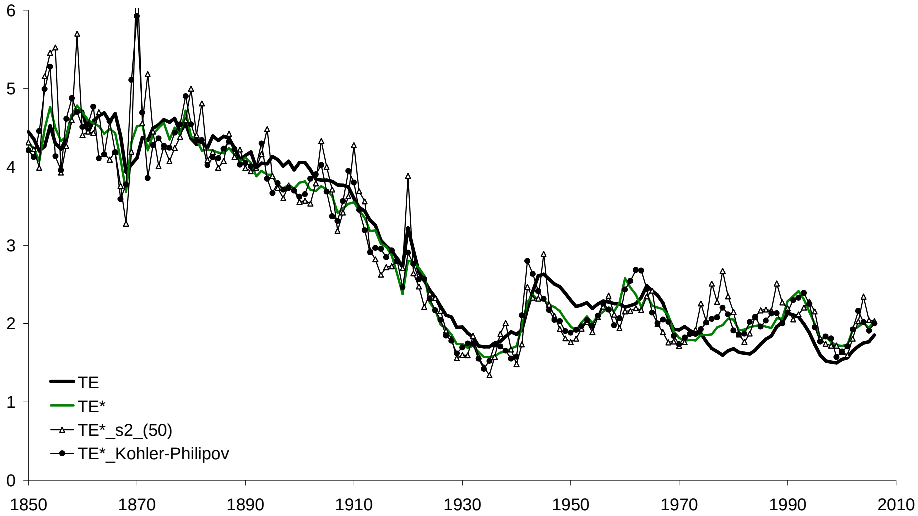
Next, in Fig. 10 we show adjustments based on the cumulated proportions of the observed unadjusted rates together with the Bongaarts-Feeney adjustments. Notoriously, the two adjustments are nearly coinciding (in fact, the difference between them is so small that one may not distinguish the two curves on Fig. 10). Such a close correspondence between the Bongaarts-Feeney adjustment and the one based on cumulated proportions of observed period rates seems to be a rule in the case of fertility. At the same time, if there were significant developments in the age structure of fertility (e.g. if postponement at some ages is combined with advancement at other), one should prefer the adjustment based on cumulated proportions rather than the one derived from the shifting hypothesis. In particular, it may be preferred due to its ability to provide an insight into the age pattern of tempo distortions. (See further down for adjusted age schedules.) One may also use the discrepancy between the two methods to indicate potential problems with the relevance of either method to the actual age pattern of tempo changes.

Figure 10 Observed period (TE) and cohort (CTE), adjusted according to the cumulated period proportions (TE*) and Bongaarts-Feeney adjusted (TE*_BF) total fertility rates for Sweden, 1850-2006. Adjustments are based on estimating derivatives by 3-year differences. Curves for the adjusted values coincide almost indistinguishably.



Source: Author’s calculation based on data from Festy (1979) for the period 1850-1960 and from the Eurostat web page (http://epp.eurostat.ec.europa.eu) for the rest of the period

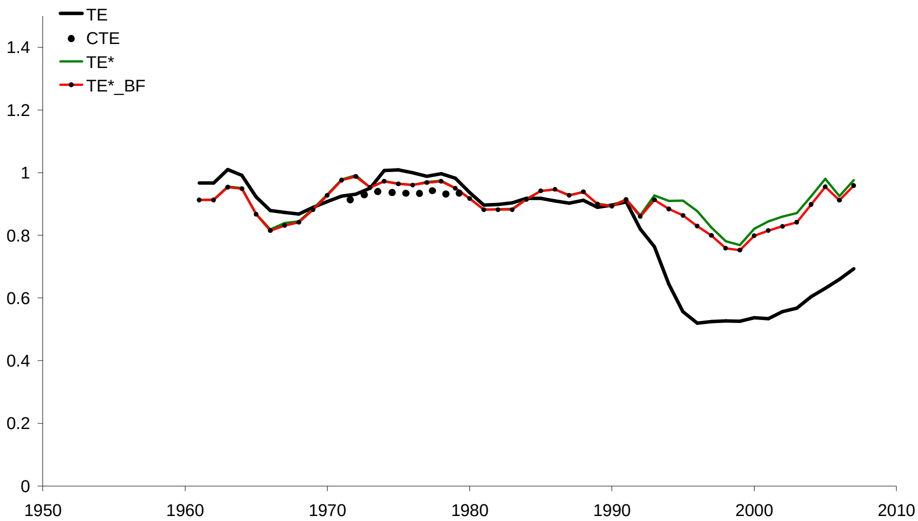
Figure 11 Observed period (TE), adjusted according to the cumulated period proportions (TE*), Kohler-Philipov adjusted and adjusted according to (50) total fertility rates for Sweden, 1850-2006. Adjustments are based on estimating derivatives by 3-year differences.



Source: Author's calculation based on data from Festy (1979) for the period 1850-1960 and from the Eurostat web page (http://epp.eurostat.ec.europa.eu) for the rest of the period

Next, we illustrate adjustments based on taking the effects of the variance into account. Adjustments based on the Kohler-Philipov iterative method (which, in addition to the variance, also takes the third moment into account) as well as those based on quadratic approximation presented in the note to Eq. (50) are shown in Fig.-11. One may note the relative similarity of both adjustments as well as their higher volatility. This volatility may be attributed to the aforementioned 'freedom of choice' of the value for the adjusted period integral quantum: due to the possibility to set the adjusted integral quantum at an almost arbitrary level, the methods easily deviate from the trend following erratic developments of the second derivative of the observed variance. (For the same reason, the methods may also significantly deviate from each other as they are based on slightly different assumptions.)

Figure 12 Observed period (TE) and cohort (CTE), adjusted according to the cumulated period proportions (TE^*) and Bongaarts-Feeney adjusted (TE*_BF) total fertility rates for first order births, Czech Republic, 1961-2007. Adjustments are based on estimating derivatives by 3-year differences. (Observed cohort rates are shifted by a period equal to the cohort mean age at childbearing.)



Source: Author's calculation based on estimates for order-specific fertility rates kindly provided by Tomas Sobotka

A final illustration of adjustments to the Total Fertility is based on fertility data[14] on first order births in the Czech Republic for 1961-2007. Bongaarts-Feeney adjustments as well as those based on cumulated period proportions of unadjusted rates are shown in Fig. 12. Again, both adjustments are rather close to each other although there are significant deviations in the period since the 1990s which was marked by strong changes in both the level and timing of fertility. Such discrepancies are an

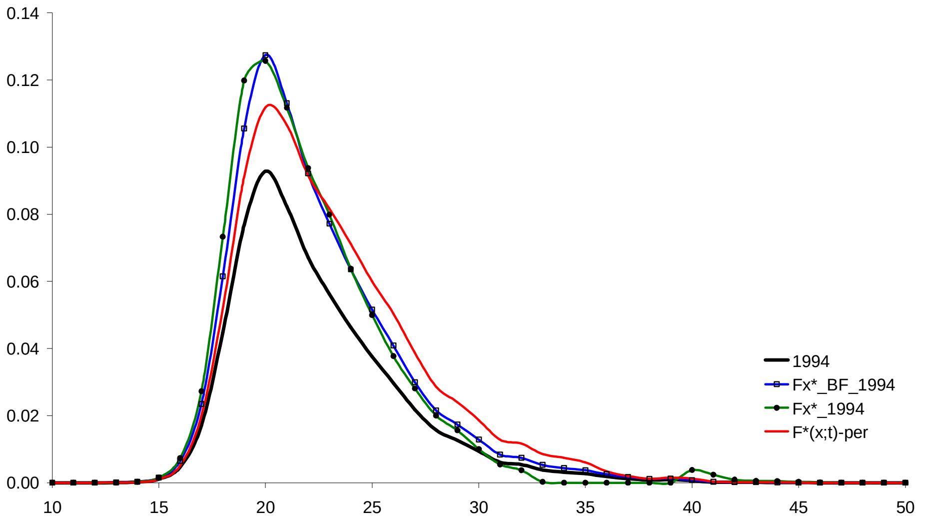
[14] I thank Tomas Sobotka for providing his estimates for order-specific fertility rates and also for important comments and suggestions.

important indication of irrelevance of the assumption underlying the BF-adjustment to the data

These relatively moderate discrepancies in fact reflect more profound differences in the adjusted age schedules of fertility. In Figs. 13 and 14 we see age schedules of first-order births fertility for the Czech Republic in 1994 and 2007: observed, adjusted according to the Bongaarts-Feeney shifting hypothesis (i.e. simply by scaling the observed schedule), adjusted according to cumulated proportions of the observed rates, and adjusted according to cumulated proportions of the true adjusted rates (41) consistent with the Bongaarts-Feeney adjustment of the Total Fertility. Similar patterns for Swedish fertility in 1945 (all birth orders combined) are presented in Fig. 15 as well as the age profile adjusted according to cumulated proportions in birth cohorts. Note the significant differences of the adjusted rates at young and advanced ages. In the first case, adjustments based on cumulated proportions show a less significant tempo effect. This is rather natural as teenage fertility, for instance, may not be reasonably assumed to postpone to elder ages; rather, it would be a better assumption to state that teenage fertility is primarily changing its quantum rather than its tempo in response to external conditions. This is especially evident in the case of Swedish fertility in 1945: it would not make any sense to assume that a considerable amount of births at young ages was postponed from the periods before (unlike other age groups where such recuperation of the previous postponement could indeed take place). At advanced ages, the true adjustment (41) consistent with Bongaarts-Feeney estimate also tends to show a less pronounced tempo effect. On the other hand, at middle childbearing ages, the true adjustment consistent with the Bongaarts-Feeney estimate shows a more significant tempo effect, a natural phenomenon for age groups in which people may change their childbearing timing more easily.

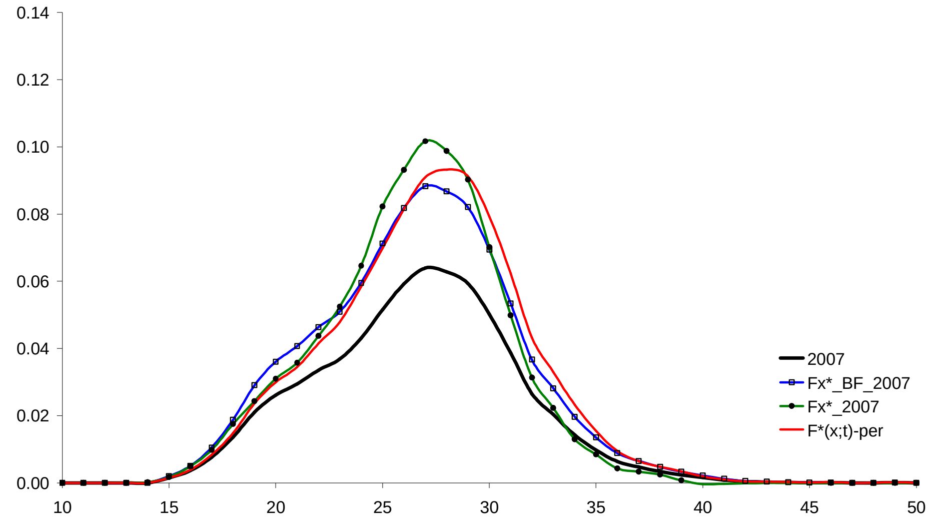
Based on the analysis presented in this paper about the case where the period- or cohort-wise proportionality of the quantum changes applies only to separate subsets of the age span, one may explicitly address the situation in which different age groups respond to external conditions in different ways. The simplest example of that kind is presented in Fig. 16, where the adjustment derived from cumulated period proportions is based on separating the age scale into two parts—below and above age 20. In both parts, cumulated proportions of observed fertility rates are used separately to adjust the rates, which implies that age 20 by itself marks the beginning of a life stage (hence, the tempo effects below and above 20 work out separately). Note that such a stylised stratification of life stages already removes the rather counterintuitive dynamics of the BF-type adjusted fertility: in fact, it does significantly decline in the early 1990s when period conditions were indeed worst, and rises afterwards.

Figure 13 Age schedules of first-order birth rates, Czech Rep., 1994: observed; proportionately scaled according to the B-F adjustment; true adjustment (41) consistent with B-F adjustment; and adjustments according to the cumulated proportions of observed period rates.



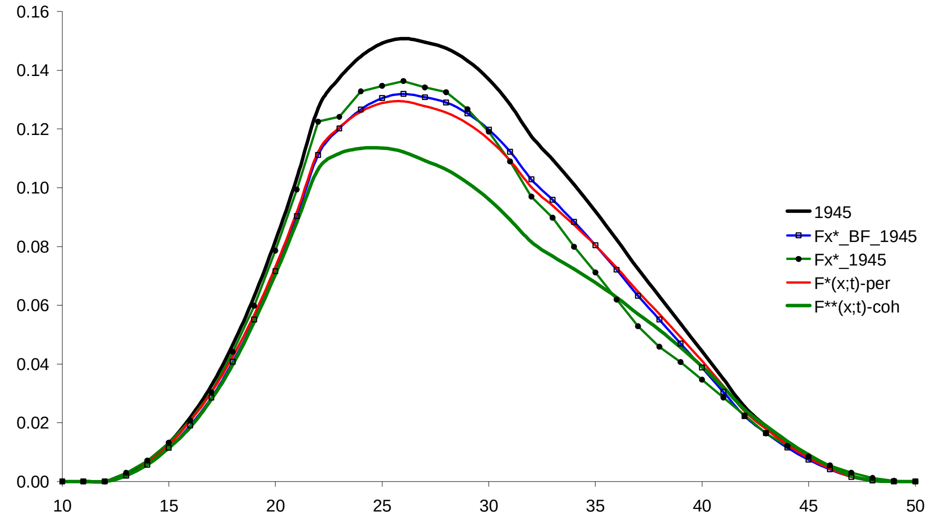
Source: Author's calculation based on estimates for order-specific fertility rates kindly provided by Tomas Sobotka

Figure 14 Age schedules of 1st order birth rates, Czech Rep., 2007: observed; proportionately scaled according to the B-F adjustment; true adjustment (41) consistent with B-F adjustment; and adjustments according to the cumulated proportions of observed period rates.



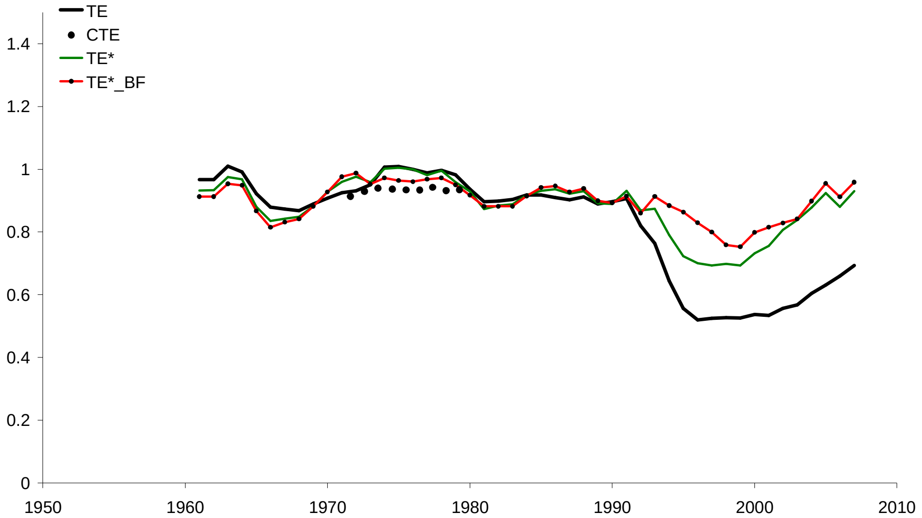
Source: Author's calculation based on estimates for order-specific fertility rates kindly provided by Tomas Sobotka

Figure 15 Age schedules of fertility rates, Sweden, 1945: observed; proportionately scaled according to the B-F adjustment; true adjustment (41) consistent with B-F adjustment; and adjustments according to the cumulated proportions of observed period and cohort rates.



Source: Author's calculation based on data from Festy (1979)

Figure 16 Observed period (TE) and cohort (CTE), adjusted according to the cumulated period proportions below and above the age 20 (TE^*) and Bongaarts-Feeney adjusted (TE*_BF) total fertility rates for first order births, Czech Republic, 1961-2007. Adjustments are based on estimating derivatives by 3-year differences. (Observed cohort rates are shifted by a period equal to the cohort mean age at childbearing.)



Source: Author's calculation based on estimates for order-specific fertility rates kindly provided by Tomas Sobotka

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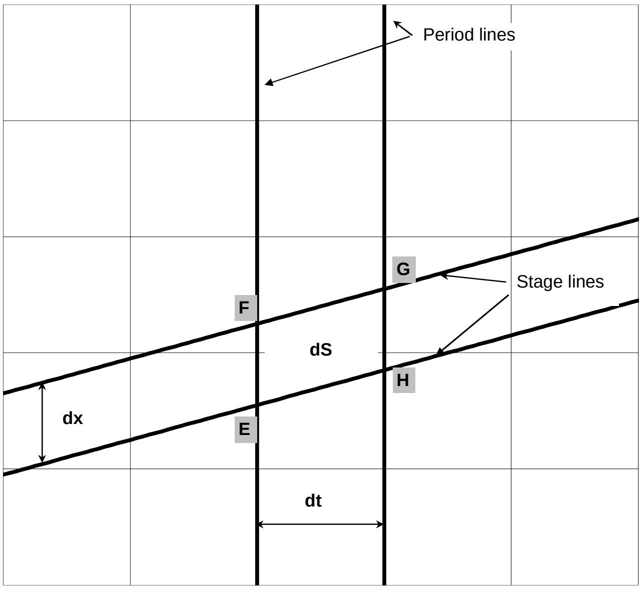
I gratefully acknowledge inspiring discussion, important comments and encouragement by John Bongaarts, important suggestions and comments by Marc Luy, Dimiter Philipov, Anatoli Yashin and Warren Sanderson, discussion and help with data from Tomáš Sobotka, and useful comments on Volterra equations by Magomed and Radmir Hubiyev. I am also indebted to thoughtful discussions of different aspects of the tempo effects at TAIG meetings at Vienna Institute of demography.

APPENDIX 1: DERIVATION OF THE FORMULAS FOR AREAS BOUNDED BY THE STAGE LINES AND BY PERIOD/COHORT LINES

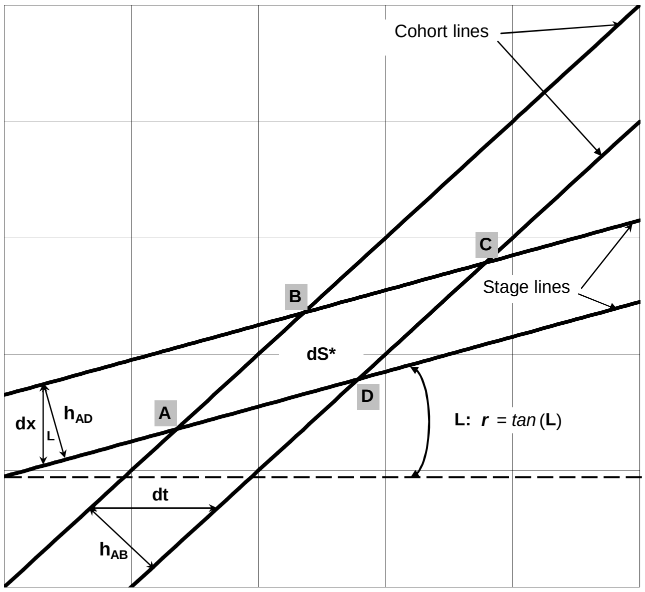
A simpler case is the case of the area between stage lines and calendar period lines (Fig. 3a reproduced below):



In this case, the area of the parallelogram **EFGH** is a product of its base **FE**, i.e. the period age increment dx corresponding to the life stage, by the vertical to the base, i.e. the distance dt between calendar time lines. Hence,

$$dS = dxdt \,. \tag{A1.1}$$

The area between the stage lines and the cohort lines demands more attention (Fig. 3b reproduced below with some additions):



In this case, the quadrangle **ABCD** has no readily known vertical that could be used in calculations. The vertical to the side **AB** may obtained from the fact that cohort lines have a slope angle $\pi/4$ to the horizontal, along which the distance between them is given by dt , i.e.

$$h_{AB} = \sin\left(\tfrac{\pi}{4}\right)dt = \frac{dt}{\sqrt{2}} . \qquad \text{(A1.1)}$$

To find the area of the quadrangle, we also need the length of the side **AB** itself. First, we find the distance between the stage lines, i.e. the vertical to the side **AD**. If L is the slope angle of the stage lines, then the same angle is formed by the calendar period lines (along which the distance between the stage lines is dx) and the vertical h_{AD} , see the figure above. Hence,

$$h_{AD} = dx \cdot \cos L . \qquad \text{(A1.2)}$$

One may also note that the angle between sides **AB** and **AD** equals $\frac{\pi}{2} - L$ and, therefore,

$$AB = \frac{h_{AD}}{\sin\left(\frac{\pi}{4} - L\right)} = \frac{\cos L}{\sin\left(\frac{\pi}{4} - L\right)} dx \,. \tag{A1.3}$$

Finally, the area may be found as the product of the side and of the vertical to it:

$$dS^* = AB \cdot h_{AB} = \frac{dxdt \cos L}{\sqrt{2} \sin\left(\frac{\pi}{4} - L\right)} = \frac{dxdt \cos L}{\sqrt{2}\left(\sin \frac{\pi}{4} \cos L - \sin L \cos \frac{\pi}{4}\right)} =$$

$$= \frac{dxdt}{1 - \tan L} \,, \tag{A1.4}$$

and the local adjustment necessary for reconciling the areas from period and cohort perspectives is given by the relation:

$$dS^* = \frac{dS}{1 - \tan L} \,. \tag{A1.5}$$

APPENDIX 2: NOTES AND ILLUSTRATION TO THE ADJUSTED INTEGRAL QUANTUM OVER FINITE-SIZE SUBSETS OF THE LEXIS SURFACE

Formally, a general formula for the integral adjustment of observations over a finite subset Ω of the Lexis surface may be obtained from (3) as

$$E^*{}_\Omega = \iint_\Omega e^*(x;t)dxdt = \iint_\Omega \frac{e(x;t)}{1-r(x;t)}dxdt = E_\Omega \iint_\Omega \frac{e(x;t)/E_\Omega}{1-r(x;t)}dxdt \text{ ,} \quad \text{(A2.1)}$$

here E_Ω and $E^*{}_\Omega$ are observed and adjusted numbers of events in the subset Ω; the last integral, which provides the integral adjustment coefficient, is a weighted average of local adjustment coefficients $\frac{1}{1-r(x;t)}$ over the subset Ω with weights proportional to the observed densities of events.

The actually observed number E_Ω would be given by integration of the unadjusted density function:

$$E_\Omega = \iint_\Omega e(x;t)dxdt \text{ .} \quad \text{(A2.2)}$$

Generally, $E^*{}_\Omega \neq E_\Omega$, except for cases where (a) there are no tempo changes and distortions; or (b) by a coincidence, all distortions within the subset Ω cancel out each other (e.g. when tempo distortions are confined to the subset Ω, and the rest of the Lexis surface is characterised by a single set of horizontal stage lines). Hence, in a realistic case of ongoing postponement or advancement of life cycle events, observed and adjusted numbers (A2.1) and (A2.2) will not be equal.

At first glance, when life stages and life cycle events analysed are linked to each other, a possible interpretation to the difference between the observed and adjusted numbers of events for the subset Ω is that the adjusted number is a number that would have been observed in the subset under no tempo changes as a similar interpretation is valid for arbitrarily small subsets addressed in the preceding Appendix and in the main text. This is not universally true, however, since assuming no tempo changes implies, in addition to the redistribution of events within the subset Ω, also bringing into the subset the events that have been postponed to the future (or advanced to the past) and taking out of the count all those events which have been advanced from the future or postponed from the past to take place in the subset but in a no tempo changes scenario would have occurred outside the subset. Such dislocations take place within cohort histories only, i.e. along the cohort lines. Hence, for subsets Ω stretching long across the cohort lines, one must be cautious in interpreting the adjusted number (A2.1) as a number that would have been observed under a no tempo changes scenario. To make this point clear, consider an extreme example where the subset is following the entire life course of a particular birth cohort. In such a case, the observed (not adjusted) densities of events being integrated over the life course must yield the true number of events in the cohort, whether or not there are tempo changes. At the same time, in the case of postponement, for example, integrating the adjusted densities over the same life course will result in a *higher* number than is actually observed in the cohort. The reason is that we integrate new (adjusted) densities over old (not transformed) durations of life stages of the cohort.

A better interpretation for finite subsets would be to avoid considering the events outside the subset and to assume a *transformation* of the boundaries of the subset itself under a no tempo changes scenario. Observed scenario and no tempo changes scenario differ with respect to the timing of life stages or, equivalently, with respect to the patterns of stage lines on the Lexis surface. When the pattern of stage lines transforms from the observed one to the hypothetical no tempo changes pattern, the subset Ω must also be transformed according to the dislocation of its content of birth cohorts' life stages. An illustration to this interpretation is presented in Figure A2.1. In the illustration, two Lexis charts are presented: (1) one that corresponds to the scenario of postponement of life cycle events by half a year each year for each age group; and (2) a second chart which corresponds to the no tempo changes scenario (we show only one of many possible hypothetical scenarios of that kind). When—due to the alteration of the scenario—stage lines are being transformed, parts of the subset Ω are also shifted along the cohort lines to a new location of a corresponding life stage. As a result, the subset shrinks significantly as at new densities of events (which are given by the adjustments considered in this paper) the same number of events will take place in less time for each of the birth cohorts. As the transformation ensures that the original and the transformed subsets contain exactly the same life stages of birth cohorts, the following equality holds between the observed and adjusted rates and the original and transformed subsets:[15]

$$E_{\Omega} = \iint_{\Omega} e(x;t)dxdt = \iint_{\Omega^*} e^*(x;t)dxdt \,. \tag{A2.3}$$

Note the integration in the second integral over the transformed subset Ω^*. The ratio of (A2.1) to (A2.2) would indicate, then, the average factor by which the subset of interest may shrink/expand under the no tempo changes scenario. (In our example it shrinks by half.) In other words, the no tempo changes scenario implies higher rates but within shorter time intervals for each of the birth cohorts involved. (Note, for instance, that in the illustration presented in Fig. A2.1. the adjusted rates are twice as high as the observed rates but they apply to the transformed subset, which is twice as small as the original one.)

The usual interpretation (that the adjusted numbers represent no tempo changes scenario for the original subset Ω) may only be validated when, for each of the birth cohorts, the densities of events are nearly equal in all the life stages confined by both Ω and Ω^* in a transformed Lexis surface (or by their pro-images in the observed Lexis surface). This condition may be considered true, for example, when for each of the birth

[15] This equation may also be considered to be an illustration to another mathematical interpretation to the adjustments in terms of the Jacobian: the adjustment factor is actually the Jacobian determinant for the transformation of the initial Lexis surface into the transformed one, which represents a no tempo changes scenario (one may also consider adjustments relating different tempo changes scenarios in the same way). See a similar interpretation in Wachter (2005), pp. 215-216.

cohorts involved, parts of the life spans of birth cohorts which fall within the subset Ω are short enough to assume densities of life cycle events in those parts, as well as in the life stages shifted outside the subset due to tempo changes, to be equal. In particular, this may be assumed in a common case where only events within a short period of time are considered.

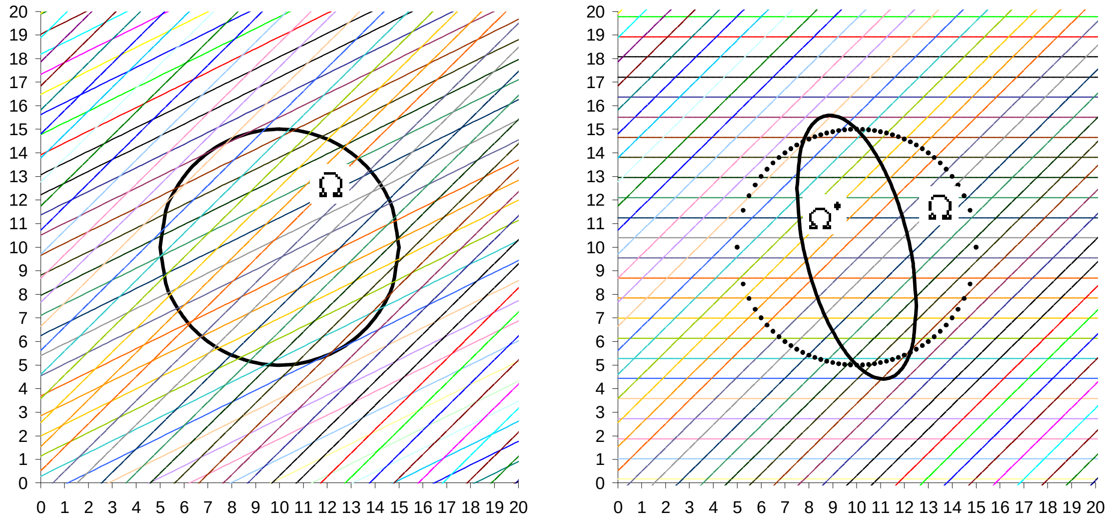
As mentioned in the main text, one may also consider the rather uncommon case where life stages and the life cycle events analysed are independent from each other, i.e. events are not changing their densities because of the changes in timing of life stages. This may apply to a situation where life stages are of demographic nature (say, linked to the timing of survival) and events of interest are of economic nature—say, personal savings, pension contributions, etc., etc. One may also consider an opposite situation where life stages do correspond closely to economic histories of cohorts when life cycle events of interest are of demographic nature (births, deaths, etc.) Such situations may be relevant to studying demographic processes in a economic context, or, vice versa—cohorts' economics in a demographic context. Clearly, in such cases adjusted densities may never be observed, even under a no tempo changes scenario, as the densities of events are determined by factors external to the tempo of cohorts' life cycles. A true interpretation then would be to say that under a no tempo changes scenario cohorts will be exposed differently to the 'external' events and, consequently, the numbers—not densities—of these events for life stages will change due to changes in the exposure time.

In any case, the ratio of the adjusted and of the observed integral quantum indicates by how much the subset of the Lexis surface is shrunk/extended due to tempo changes (weighting transformations by observed densities of events in respective life stages):

$$\frac{\|\Omega^*\|}{\|\Omega\|} = \frac{E^*{}_\Omega}{E_\Omega} = \iint_\Omega \frac{e(x;t)/E_\Omega}{1-r(x;t)}\,dxdt\ , \tag{A2.4}$$

Figure A2.1 Transformation of subsets of the Lexis surface according to the change in the stage lines' pattern

Actual pattern: postponement by 0.5 year each year. The subset Ω is a circle shown by the bold line	Hypothetical no-tempo-changes pattern. The transformation Ω^* of the original subset is shown by the bold line, while the original subset is marked by dots



Source: Author's simulations

APPENDIX 3: ON THE RESOLVABILITY OF THE INTEGRAL EQUATION (41) FOR THE ADJUSTED QUANTUM IN CASE OF PERIOD-WISE PROPORTIONAL CHANGES

First, we demonstrate how the equation may be transformed into a form of Volterra equations. Consider the equation with explicitly used upper limit of life span:

$$\int_0^X \frac{e(y;t+y)}{TE^*(t+y)}dy = 1,\ t \in [0,T]. \quad \text{(A3.1)}$$

Since observations start with some calendar period or cohort, Eq. (A3.1) must be supplemented by boundary conditions at the beginning of the observation period.

Using substitutions $u = t + y$, $\tau = t + X$, $f(u) = \frac{1}{TE^*(u)}$, we see that (A3.1) transforms to:

$$\int_0^\tau e(u - \tau + X;u) f(u)du = 1,\ \tau \in [X, T + X]. \quad \text{(A3.2)}$$

(For the sake of simplicity, we keep the lower bound of integration at value 0 by formally assuming zero density of life cycle events for negative ages.) When $\tau < X$ (i.e. for incomplete cohorts in the beginning of the observation period), one may use the boundary conditions imposed on the stage function:

$$\int_0^\tau e(u - \tau + X;u) f(u)du = s(\tau;0),\ \tau \in [0, X]. \quad \text{(A3.3)}$$

i.e.

$$\int_0^\tau e(u - \tau + X;u) f(u)du = g(\tau) = \begin{cases} s(\tau;0),\ \tau < X \\ 1,\ \tau \ge X \end{cases} \quad \text{(A3.4)}$$

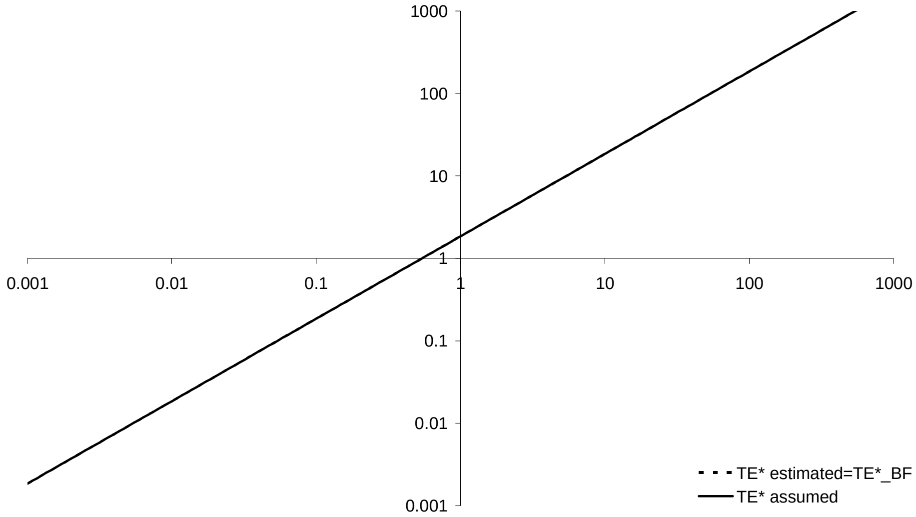
(Note, $g(0) = s(0;0) = 0$.)

Eq. (A3.4) is a form of Volterra integral equations of the first kind. Its important feature is that its kernel $K(\tau,u) = e(u - \tau + X;u)$ turns zero together with all its derivatives when its variables coincide: $K(u,u) = e(X;u) = 0$ (at the upper limit of the life span density of life cycle events turns zero). Such integral equations are known to be incorrect and their solution requires using regularisation conditions or a priori information (e.g. Tikhonov and Arsenin 1979) In our case, we demonstrate in the main text that such conditions may be imposed by assuming the level of the adjusted integral quantum.

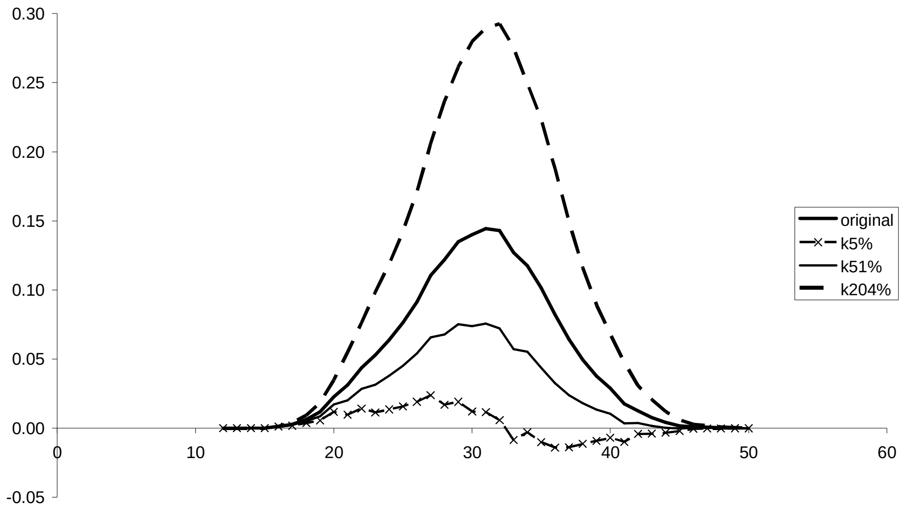
Another interpretation to the incorrectness of Eq. (A3.4) is that its solutions are very sensitive to data errors and incorrectness of the model assumption, which in practice prevents from robustly solving the equation.

To illustrate the looseness of the restriction imposed by the equation, or the freedom of choice of the level of the adjusted integral quantum, we conducted the following numerical experiments. Taking the observed age profile of Swedish fertility in 2006, considering it as non-distorted, we form the boundary condition on the stage function (A3.3) derived from cumulated proportions of observed rates. Then, we extend the same profile into 2007 and try different adjustments for 'tempo distortions'. A reasonable 'true' adjustment would indeed assume no distortions as the fertility rates remained constant in our example. However, to check the sensitivity of solutions to the equation (A3.1), we formally try a wide range of different assumptions about the adjusted Total Fertility in 2007. We vary the assumed adjusted level from 0.001 of the observed TFR to 1000 times the observed TFR (which is 1.85), a range that comprises all realistic situations. For each of the assumed adjusted TFR we compute a stage function from (39) and calculate the adjusted profile of fertility rates (41). Finally, we compare the initially assumed adj. TFR to the one eventually obtained and also examine the age profile of adjusted fertility rates. Hence, one may interpret our calculations as a first step in an iterative procedure used to 'solve' the equation we consider. We also conduct all these calculations with an additional correction imposed on the calculated stage function, which ensures that this function is monotonically increasing to reach the eventual level of 1. To ensure that, we estimate values of the function at ages 40+ from remaining proportions of the observed fertility rates and not from (39) (for practical applications, this would mean that at advanced ages we relax the proportionality assumption in order to keep the stage function well-shaped; given small proportion of fertility at ages 40+, such an adjustment would not affect the overall adjustment much in practical cases).

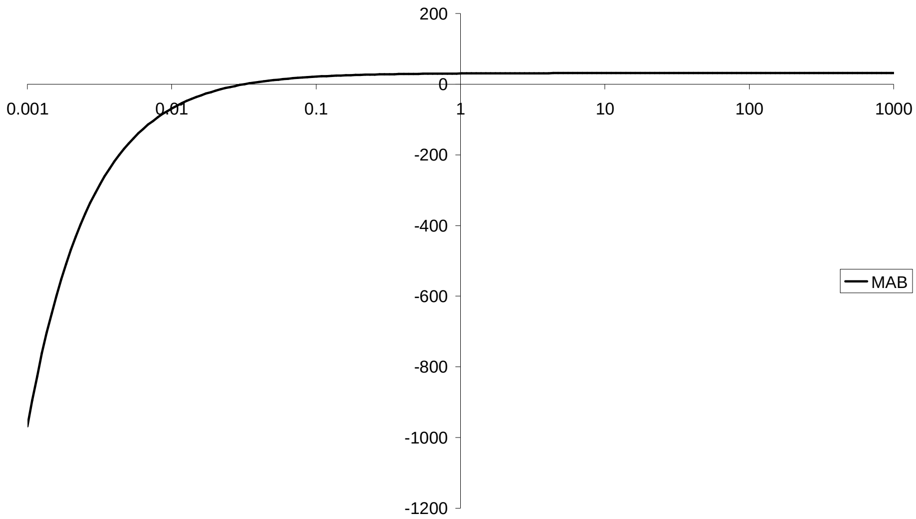
The following figure presents the eventually estimated adjusted TFR as a function of the one originally assumed in the case where no corrections are applied to the stage function (note logarithmic scales):



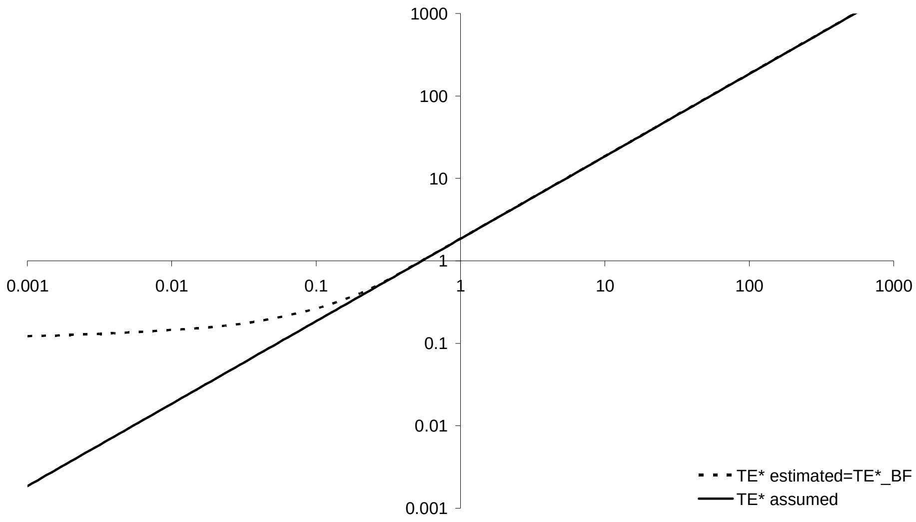
Note that both the ‘assumed’ and ‘estimated’ adjusted TFRs are identical, which, among other implications, shows that iterations cannot help in solving the equation (A3.1). However, the stage function and therefore the adjusted age pattern of fertility are rather artificial in cases of extreme ‘assumptions’ about the adjusted TFR. The following figure shows the adjusted age patterns of fertility for some of the choices for the ratio of the assumed adj. TFR to the one actually observed (original profile as well as those for assumptions at levels 5%, 51%, and 204% of the observed TFR are shown):



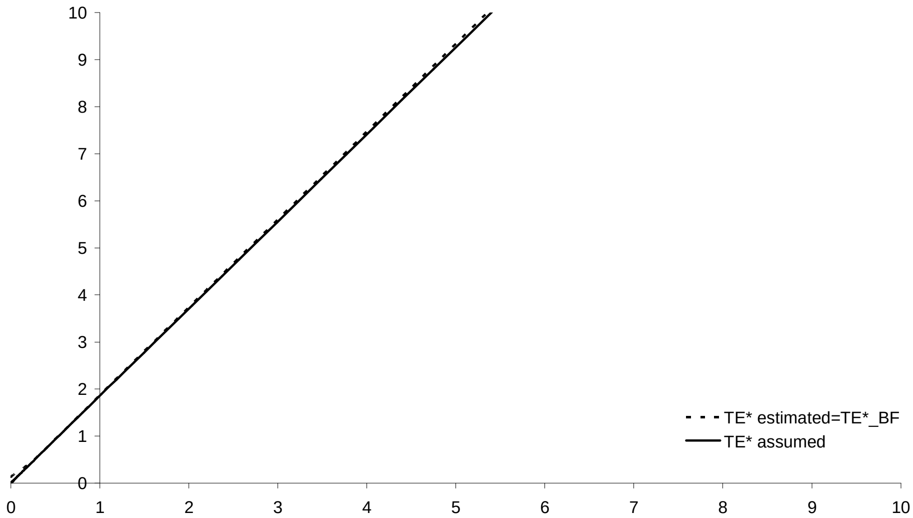
Note that only in case of extremely low assumptions about the adjusted TFR some of the adjusted fertility rates turn negative; in other cases there is no problem with finding a reasonable age profile of adjusted fertility that fits the assumed level of the adjusted TFR. Also note that, following from the analysis presented in this paper, each of the arbitrary choices for the adjusted TFR is compatible with the Bongaarts-Feeney-type method where it is based on the change of the adjusted mean age at childbearing: the adjusted mean age simply changes in such a way that the Ryder-Bongaarts-Feeney adjustment exactly coincides with the assumed adjusted TFR. To see that, one may examine, how the adjusted mean age at childbearing changes according to different assumptions about the adjusted TFR (note negative values for extremely low assumptions):



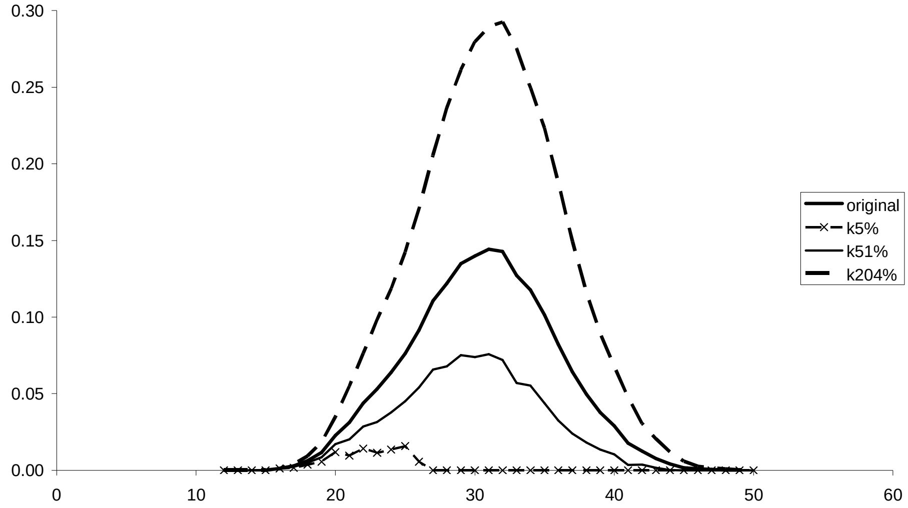
Hence, for high assumptions about the adjusted fertility there are no problems in finding an adjusted profile, which would ‘prove’ any level of the adjustment. However, for low levels of the adj. TFR one should correct the stage function in the aforementioned way, so that the adjusted fertility profiles look realistic. The following figure presents estimated and assumed adjusted TFR pattern for that case (note the logarithmic scale, which exaggerates discrepancies at small levels):



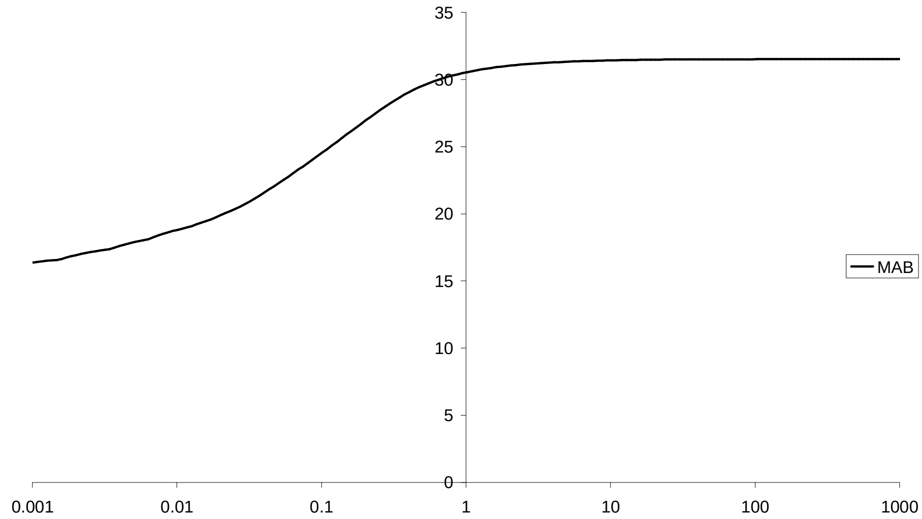
Note that the assumed and estimated adj. TFR differ considerably only when the assumed level is less then about 10% of the actual value. This is more evident from the following figure, which represents the same results, but for a smaller range of the assumed adj. TFR and without logarithmic scale:



In any case, the difference between the assumed and estimated levels is negligible compared to the observed TFR (1.85). Meanwhile, age patterns of the fertility rates are realistic (note that all rates are non-negative):



Similarly, the adjusted mean age changes rather realistically and ensures that the Ryder-Bongaarts-Feeney-type adjustment will coincide with the assumed adjusted TFR:



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