Ramsey-Skiba utility maximization. A classical model with incomplete theory

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Outline

1. Utility Maximization. The Ramsey-Skiba model
   - Model description
   - Dynamic Programming for RS problems
   - An existence theorem for RS problems

Economic Growth: the Ramsey-Skiba model

Infinite horizon utility maximization problem for an endogenous growth model with **convex-concave** production function, introduced first by Skiba (1978) and studied by various authors (e.g. Nishimura et al. (1983), Askenazy - Le Van (1999), Santos et al. (1999), Fiaschi and Gozzi (2009)).

The model generalizes the original concave-dynamics model by Ramsey (1928).

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- Important part of economic growth literature
- Lack of generality in recent works facing convex-concave dynamics

In particular, what is the state of the art for the existence question?

- Seemingly, the question is (positively) answered only for the **concave dynamics** case (Ekeland 2010, Asheim and Ekeland 2016)
- In latter works, the optimum is obtained via a feedback relation; this approach requires to assume $\mathscr{C}^2$ regularity on the admissible states and on the value function
- For the same case, there is also a direct method in Freni, Gozzi and Salvadori 2010

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Qualitative features of the model

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- $x(t) =$ capital and $c(t) =$ consumption rate, at time $t \geq 0$
- $u(c) =$ instantaneous utility from consumption, increasing and concave
- $F(x) =$ per capita production function increasing, **not concave**
- **state constraint:** $x(t) \geq 0$, for $t \geq 0$

Quantitative description of the model

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- Let $x(\cdot;x_0,c)$ be the unique solution of
$$\begin{cases} \dot{x}(t) = F(x(t)) - c(t) & t \geq 0 \\ x(0) = x_0 \geq 0 \end{cases}$$
- Goal: maximize
$$J(x_0;c) := \int_0^{+\infty} e^{-\rho t} u(c(t))\,dt$$
for all $c \in \Lambda(x_0)$, where
- $\Lambda(x_0) := \{c \in L^1_{loc}([0,+\infty),\mathbb{R}) / c \geq 0 \text{ a.e.}, x(\cdot;x_0,c) \geq 0\}$

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Main assumptions

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- $J(x_0; c) := \int_0^{+\infty} e^{-\rho t} u(c(t))\, dt \qquad c \in \Lambda(x_0)$
- $\begin{cases} \dot{x}(t) = F(x(t)) - c(t) & t \geq 0 \\ x(0) = x_0 \end{cases}$

$$\begin{cases} F \in \mathscr{C}^1(\mathbb{R}, \mathbb{R}) \\ F' > 0 \text{ in } \mathbb{R} \\ F(0) = 0 \\ \lim_{x \to +\infty} F(x) = +\infty \\ \lim_{x \to +\infty} F'(x) =: L > 0 \\ F \text{ is convex in } [0, \overline{x}] \\ F \text{ is concave in } [\overline{x}, +\infty) \end{cases}$$

$$\begin{cases} u \in \mathscr{C}^2((0, +\infty), \mathbb{R}) \\ u \in \mathscr{C}^0([0, +\infty), \mathbb{R}) \\ u(0) = 0, \ \lim_{x \to +\infty} u(x) = +\infty \\ u' > 0,\ u'' < 0 \\ \lim_{x \to 0^+} u'(x) = +\infty, \ \lim_{x \to +\infty} u'(x) = 0 \\ \lim_{t \to +\infty} e^{\varepsilon_0 t} e^{-\rho t} u\left(e^{(L+\varepsilon_0)t}\right) = 0, \\ \text{for some } \varepsilon_0 > 0 \end{cases}$$

Finiteness of the value function

The value function is $V(x_0) := \sup_{c \in \Lambda(x_0)} J(x_0; c) \quad \forall x_0 \geq 0.$

Note: Condition

$$\exists \varepsilon_0 > 0 : \lim_{t \to +\infty} e^{\varepsilon_0 t} e^{-\rho t} u\left(e^{(L+\varepsilon_0)t}\right) = 0$$

is an *if and only if* condition for the value function to be finite for every $x_0 \geq 0$ in the case:

$$u(c) = c^{1-\sigma}, \quad F(x) = Lx$$

Goals of the analysis

Ultimate goal of the analysis

- Properties of optimal paths (x^*, c^*), improving the previous results.
  For example: conditions for the monotonicity of c^* are not known, while monotonicity of x^* is proved in Askenazy - Le Van and in Nishimura et al.

Main Tool

- Deep study of the properties of the value function, in particular $\mathscr{C}^1$ regularity and presence of singularities in the first derivative.

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  For example: conditions for the monotonicity of c^* are not known, while monotonicity of k^* is proved in Askenazy - Le Van and in Nishimura et al.

To do this, the research path is:

- Existence of optimal controls
- Basic properties of the value function (behaviour at 0^+ and $+\infty$, "strong" monotonicity, local Lipschitz continuity)
- The value function solves the Hamilton-Jacobi-Bellman equation, in the (bilateral) viscosity sense
- Pontryagin Maximum Principle

Results up to now

Up to now, the following results have been proven

- Existence of the optimal controls
- Finiteness, boundary behaviour, Lipschitz-continuity, "strong" monotonicity of the value function
- Dynamic programming: the value function solves the Hamilton-Jacobi-Bellman equation, in the viscosity sense (bilateral solution)
- Preliminary work to first order necessary conditions for optimality (Pontryagin Maximum Principle)

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HJB Equation for state-constrained problems

The **Hamilton-Jacobi-Bellman** (HJB) equation is a classical **necessary condition** that a function of the initial state is supposed to verify in order to be the **value function** of the problem.

In an infinite horizon problem HJB takes the form:

$$\rho v(x) + H(x, \nabla v(x)) = 0 \quad x \in \Omega,$$

- $e^{-\rho t}$ is the discount factor in $J(x;\cdot)$
- state equation is autonomous
- Ω contains admissible initial states

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Does the following hold for $\mathrm{v} = V$ (the value function)?

$$\rho \mathrm{v}(x) + H(x, \nabla \mathrm{v}(x)) = 0 \quad x \in \Omega.$$

- The value function V is not necessarily $\mathscr{C}^1 \implies$ search for weaker solutions, i.e. **viscosity solutions** (Crandall and Lions).

What is the general form of the equation, or the "right notion" of weak (viscosity) solution if H derives from a **state-constrained** OC problem?

- The question has been faced (with different answers) in works such as Soner (1986), Capuzzo-Dolcetta and Lions (1990) and Ishii and Koike (1996).

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HJB Equation for state-constrained problems

$$\rho \mathrm{v}(x) + H(x, \nabla \mathrm{v}(x)) = 0 \quad x \in \Omega,$$
$$H(x,p) = -\sup_{c \in \text{control space}} \{\langle b(x,c), p\rangle + L(x,c)\}$$

- If an admissible trajectory $x(\cdot;, c)$ hits $\partial\Omega$ in x_0 at time t_0 then
$$\langle b(x_0, c(t_0)), \nu_\Omega(x_0)\rangle \le 0 \quad (\nu_\Omega \text{ exterior normal}).$$
- Let x_0 be *local maximum* point of $V - \varphi$ in case V is $\mathscr{C}^1$ value function:
$$\rho \mathrm{v}(x_0) + H(x_0, \nabla\varphi(x_0)) = \rho \mathrm{v}(x_0) + H\left(x_0, \nabla V(x_0) - \underbrace{\lambda}_{\ge 0} \nu_\Omega(x_0)\right)$$
$$\ldots \le \rho \mathrm{v}(x_0) + H(x_0, \nabla V(x_0))$$
$$= 0.$$

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$$\rho \mathrm{v}(x) + H(x, \nabla \mathrm{v}(x)) = 0 \quad x \in \Omega,$$
$$H(x,p) = - \sup_{c \in \text{control space}} \{\langle b(x,c), p\rangle + L(x,c)\}$$

- Thus if $x_0 \in \partial\Omega$:

$$\langle b(x_0, c(t_0)), \nu_\Omega(x_0)\rangle \leq 0 \quad (\nu_\Omega \text{ exterior normal})$$
$$\Longrightarrow \rho \mathrm{v}(x_0) + H(x_0, \nabla \varphi(x_0)) \leq 0$$

whenever x_0 is a *local maximum* point of $V - \varphi$ (V, φ in $\mathscr{C}^1$).

Definition (Soner)

$\mathrm{v} : \overline{\Omega} \to \mathbb{R}$ is a *constrained viscosity solution* of HJB (for maximum value problems) $\iff$ v is subsolution in $\overline{\Omega}$ and supersolution on Ω.

HJB Equation for state-constrained problems

Theorem (Soner, 1986)

Assume that the value function of the problem

$$\text{maximize } J(x_0; c) = \int_0^{+\infty} e^{-\rho t} L(x(t), c(t))\, dt$$
$$\text{subject to } \dot{x}(t) = b(t, x(t), c(t)),\ x(0) = x_0$$
$$(c(t), x(t)) \in K \times \overline{\Omega}$$

is in $BUC(\overline{\Omega})$*. Then it is the unique constrained visc. solution of*

$$\rho \mathrm{v}(x) + H(x, \nabla \mathrm{v}(x)) = 0 \quad x \in \Omega,$$
$$H(x,p) = -\sup_{c \in K} \{\langle b(x,c), p\rangle + L(x,c)\}$$

in $BUC(\overline{\Omega})$*, if, in particular,* K *is compact and* b*,* L *are bounded.*

HJB Equation for RS problems

The HJB equation for RS problems is:

$$\rho \mathrm{v}(x) + H\left(x, \mathrm{v}'(x)\right) = 0 \quad x > 0$$

where

$$H(x,p) := -\sup_{c \geq 0} \{[F(x) - c] \cdot p + u(c)\}$$

$$H(x,p) > -\infty \iff p > 0$$

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$$\rho \mathrm{v}(x) + H\left(x, \mathrm{v}'(x)\right) = 0 \quad x > 0 \tag{1}$$
$$H : [0, +\infty) \times (0, +\infty) \to \mathbb{R}$$

- v is a viscosity solution of 1 if and only if, for any $x_0 > 0$, $\varphi \in \mathscr{C}^1\left(I\left(x_0\right)\right)$ we have:

x_0 is a local maximum of $\mathrm{v} - \varphi \implies \rho \mathrm{v}(x_0) + H\left(x_0, \varphi'(x_0)\right) \leq 0$

x_0 is a local minimum of $\mathrm{v} - \varphi \implies \rho \mathrm{v}(x_0) + H\left(x_0, \varphi'(x_0)\right) \geq 0$

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$$H : [0, +\infty) \times (0, +\infty) \to \mathbb{R}$$

A candidate solution v must be such that, in the above cases, $\varphi'(x_0) > 0$.

- Among the other properties of the value function, we prove that

$$\frac{V(x_0 + h) - V(x_0)}{h} \geq C(x_0) > 0$$

for every $x_0 > 0$ and every h sufficiently small (either positive or negative).

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$$\forall h \in (-\delta, \delta) : \frac{V(x_0 + h) - V(x_0)}{h} \geq C(x_0) \implies \varphi'(x_0) > 0$$

for every $\varphi \in \mathscr{C}^1(I(x_0))$ such that x_0 is an extremum point of $V - \varphi$.

We prove this result - say $V \in \mathscr{C}^+(0, +\infty)$ - using the *localization lemma* (see next section).

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Remark:

- The implication when x_0 is a local maximum requires the use of the incremental quotient of V for $h < 0$.
- As far as the quotients of V are concerned, the case $h < 0$ is not symmetric to the case $h > 0$.
- This is linked to statement the before mentioned localization lemma

HJB Equation for RS problems

Bellman equation

- Before proving that V solves (HJB) in the viscosity sense, we need to establish that it solves the *Bellman Equation* (Dynamic Programming Principle).

HJB Equation for RS problems

Bellman equation

Theorem

For every $\tau > 0$, the value function $V : [0, +\infty) \to \mathbb{R}$ satisfies the functional equation:

$$\forall x_0 \geq 0 : \mathrm{v}(x_0) = \sup_{c \in \Lambda(x_0)} \left\{ \int_0^\tau e^{-\rho t} u(c(t))\, dt + e^{-\rho \tau} \mathrm{v}(x(\tau; x_0, c)) \right\}$$

Corollary

Let $x_0 \geq 0$, $c^* \in \Lambda(x_0)$. Thus the following are equivalent:
i) c^* is optimal at x_0
ii) For every $\tau > 0$:

$$V(x_0) = \int_0^\tau e^{-\rho t} u(c^*(t))\, dt + e^{-\rho \tau} V(x(\tau; x_0, c^*))$$

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Corollary

Let $x_0 \geq 0$, $c^* \in \Lambda(x_0)$. If c^* is optimal at x_0, then for every $\tau > 0$, $c^*(\cdot + \tau)$ is admissible and optimal at $x(\tau; x_0, c^*)$.

HJB Equation for RS problems

Theorem

Let $x_0 > 0$ *and* $(c_T)_{T>0} \subseteq \Lambda(x_0)$ *satisfying:*

$$\|c_T\|_{\infty,[0,T]} \leq N(x_0, T) \quad \forall T > 0.$$

where N *is the function defined in the localization lemma. Hence*

$$\forall T \in [0,1] : \forall t \in [0, T] : |x(t; x_0, c_T) - x_0| \leq Te^{\bar{M}t}[F(x_0) + N(x_0, 1)].$$

In particular $x(T; x_0, c_T) \to x_0$ *as* $T \to 0$.

This is used to prove that V is a viscosity supersolution of HJB

But in order that we can reduce to such families of admissible controls, the *localization lemma* must be used (again!)

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In particular $x(T;x_0,c_T) \to x_0$ as $T \to 0$.

Thanks to the *localization lemma* the above estimate can be exploited to obtain:

Corollary

The value function is a viscosity solution to HJB in $\mathscr{C}^+(0,+\infty)$.

HJB Equation for RS problems

Indeed it is a fact that:

Theorem

The value function V is a constrained viscosity solution to HJB (in the sense of Soner) in $\mathscr{C}^+([0,+\infty))$, meaning that:
i) V is a viscosity solution in $(0,+\infty)$
ii) The subsolution condition for V in 0 is meaningful and satisfied

Reverse HJB Equation for RS problems

$$\rho \mathrm{v}(x) + H(x, \nabla \mathrm{v}(x)) = 0 \quad x \in \Omega.$$

x_0 is a local maximum of $\mathrm{v} - \varphi \implies \rho \mathrm{v}(x_0) + H(x_0, \nabla \varphi(x_0)) \leq 0$

x_0 is a local minimum of $\mathrm{v} - \varphi \implies \rho \mathrm{v}(x_0) + H(x_0, \nabla \varphi(x_0)) \geq 0$

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We consider

$$\begin{cases} \dot{y}(t) = F(y(t)) - c & t \leq 0 \\ y(0) = y_0 \end{cases}$$

$$\begin{cases} \dot{y}(t) = F(y(t)) - c^*(t+T) & t \in [-T, 0] \\ y(0) = y_0 \end{cases}$$

Then we prove a "backward" Dynamic Programming Principle for optimal controls at optimal points, and we obtain..

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Reverse HJB Equation for RS problems

$$-\rho \mathrm{v}(x) - H\left(x, \mathrm{v}'(x)\right) = 0 \quad x \in (0, +\infty) \tag{2}$$

Theorem

If $x_0 = x(T; y_0, c^*) > 0$ *with* c^* *optimal at* y_0*, then:*

$\forall \tau \in (0, T]$:

$$V(x_0) = -\int_0^{\tau} e^{\rho t} u(c^*(-t+T))\, dt + e^{\rho\tau} V(\theta(-\tau; x_0, c^*(\cdot + T))).$$

Consequently, the value function V *is a viscosity supersolution of the "reverse" HJB equation (2) at optimal points and a viscosity subsolution in* $(0, +\infty)$*.*

Outline

1. Utility Maximization. The Ramsey-Skiba model
   - Model description
   - Dynamic Programming for RS problems
   - An existence theorem for RS problems

Infinite horizon optimal control problems

$$\text{maximize } J(x_0;c) = \int_0^{+\infty} e^{-\rho t} L(t,x(t),c(t))\,\mathrm{d}t$$

$$\text{subject to} \begin{cases} \dot{x}(t) = b(t,x(t),c(t)) & t \geq 0 \\ x(0) = x_0 \end{cases}$$

As far as we know, **no general existence theory** satisfactorily covering the case of **unbounded control space** (and non-concave space dynamics for scalar problems) is available (see Seierstad and Sydsæter 1987, Carlson Haurie and Leizarowitz 1991, Balder 1993).

Thus, we developed an original method applied to three classes of problems, deriving from two classical economic models:

- Ramsey-Skiba model
- monotone Shallow lake model
- non-monotone Shallow lake model.

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Problems from economics

The latter are *Infinite horizon* optimal control problems, in which controls may be *unbounded* in finite time intervals, and the dynamics has *convex-concave* behaviour.

Existence of solutions for such problems is not covered by the literature. But what are the main technical difficulties?

- The infinite horizon setting plus the unboundedness of the controls prevent from having good *a priori* estimates, thus the application of any compactness results is not straightforward
- Non-concavity in space of the dynamics
- Possible presence of a state constraint (Ramsey-Skiba)

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Step 1: Localization lemma

$$J(c) = \int_0^{+\infty} e^{-\rho t} u(c(t))\,dt$$

Theorem

There exists a function $N : [0,+\infty)^2 \to (0,+\infty)$, increasing in both variables, such that:
for every $(x_0, T) \in [0,+\infty)^2$ and every admissible c, there exists an admissible c^T satisfying

$$J\left(x_0; c^T\right) \geq J\left(x_0; c\right)$$

$$c^T = c \wedge N\left(x_0, T\right) \text{ almost everywhere in } [0,T]$$

In particular, c^T is bounded above, in $[0,T]$, by a quantity which does not depend on the original control c, but only on T and on the initial state x_0.

Note: This was useful also in Dynamic Programming.

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Step 2: double-sequence diagonalization

- For $x_0 > 0$ fixed, take a maximizing sequence

$$\lim_{n \to +\infty} J(x_0; c_n) = V(x_0)$$

- Use recursively the localization lemma and then the Dunford-Pettis criterion: at time $T+1$, start by the control obtained at time T:

$$(c_n)_n \overset{Loc.Lemma}{\dashrightarrow} (c_n^{\mathbf{1}})_n \overset{D.P.}{\supseteq} (\overline{c}_n^{\mathbf{1}})_n \overset{Loc.Lemma}{\dashrightarrow} (c_n^{\mathbf{2}})_n \overset{D.P.}{\supseteq} (\overline{c}_n^{\mathbf{2}})_n \dashrightarrow \dots$$

- $\overline{c}_n^{\mathbf{1}} \rightharpoonup c^{\mathbf{1}}$ in $L^1([0,1])$, $\quad \overline{c}_n^{\mathbf{2}} \rightharpoonup c^{\mathbf{2}}$ in $L^1([0,2])\dots$
- We still have $J(x_0; \overline{c}_n^{T}) \to V(x_0)$ for every $T > 0$.

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$$\cdots \dashrightarrow \left(c_n^{\mathbf{T}}\right)_n \overset{D.P.}{\supseteq} \left(\overline{c}_n^{\mathbf{T}}\right)_n \overset{Loc.Lemma}{\dashrightarrow} \left(c_n^{\mathbf{T}+1}\right)_n \overset{D.P.}{\supseteq} \left(\overline{c}_n^{\mathbf{T}+1}\right)_n \dashrightarrow \dots$$

The sequence $\left(\overline{c}_n^T\right)$ is extracted from $\left(c_n^T\right)$, but

a priori, $\left(c_n^{T+1}\right)$ is not extracted from $\left(\overline{c}_n^T\right)$.

- Using the monotonicity of the function

$$T \to N(x_0, T)$$

it is possible to show that $\left(c_n^{T+1}\right)$ coincides, a.e. in $[0, T]$, with a subsequence of $\left(\overline{c}_n^T\right)$.

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$$\gamma = \sum_{T=1}^{\infty} c^T \chi_{[T-1,T]} \text{ (sub-optimal)}$$

$$(\gamma_n)_n = \text{first approximating sequence.}$$

- $(\gamma_n)_{n\geq 0} \subseteq \Lambda(x_0)$, $\gamma \in L^1_{loc}([0,+\infty))$ such that

$$J(x_0;\gamma_n) \to V(x_0)$$
$$\gamma_n \rightharpoonup \gamma \text{ in } L^1([0,T]) \quad \forall T > 0$$
$$0 \leq \gamma_n, \gamma \leq N(x_0,T) \quad \forall T, \text{ a.e. in } [0,T], n \geq T$$

Note: $(\gamma_n)_n$ coincides a. e. in $[0,T]$ with a subsequence of $\left(\overline{c}_n^T\right)_n$. Thus both $(\gamma_n)_n$ and γ inherit the local boundedness properties of sequences $\left(\overline{c}_n^T\right)_n$, $T \in \mathbb{N}$.

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$\gamma =$ sub-optimal control, $(\gamma_n)_n =$ first approximating sequence.

- $(\gamma_n)_{n\geq 0} \subseteq \Lambda(x_0)$, $\gamma \in L^1_{loc}([0,+\infty))$ such that

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$$\implies 0 \leq x_{\gamma_n} \overset{n\to+\infty}{\longrightarrow} x_\gamma \text{ pointwise in } [0,+\infty).$$

In particular, this guarantees the admissibility of γ .

Step 3: simple diagonalization and functional convergence

$$J(c) = \int_0^{+\infty} e^{-\rho t} u(c(t))\,\mathrm{d}t = \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(c(\sigma))\,\mathrm{d}\sigma \mathrm{d}t$$

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Consider the composition of u and the γ_n's:

$$f_n = u(\gamma_n)$$

- The sequence $(f_n)_n$ is locally uniformly bounded
- Standard diagonalization: $(f_{n,n})_n$ extracted from $(f_n)_n$ such that

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$$u(\gamma_{n,n}) \rightharpoonup f \text{ in } L^1([0,T]) \quad \forall T > 0$$

- Inversion:

$$c^* := u^{-1}(f)$$

- Hence by construction, in $L^1([0,T])$ for every $T > 0$:

$$\gamma_n \rightharpoonup \gamma$$
$$u(\gamma_{n,n}) \rightharpoonup u(c^*)$$

- Pointwise in $[0, +\infty)$:

$$x_{\gamma_n} \to x_\gamma.$$

Step 3: simple diagonalization and functional convergence

- In $L^1([0,T])$ for every $T > 0$:

$$\begin{aligned} \gamma_n &\rightharpoonup \gamma \\ u(\gamma_{n,n}) &\rightharpoonup u(c^*) \end{aligned}$$

- Moreover it can be proven that

$$x^* \geq x_\gamma \text{ pointwise in } [0, +\infty) \tag{3}$$

using concavity arguments.
In particular, relation (3) implies that c^* if admissible.

Step 3: simple diagonalization and functional convergence

$$u(\gamma_{n,n}) \rightharpoonup u(c^*) \quad \text{in } L^1([0,T]),\ \ T>0$$

Convergence of the functionals:

$$\begin{aligned} V(x_0) &= \lim_{n\to+\infty} J(x_0;\gamma_{n,n}) \\ &= \lim_{n\to+\infty} \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(\gamma_{n,n}(s))\, ds dt \\ &= \rho \int_0^{+\infty} e^{-\rho t} \lim_{n\to+\infty} \int_0^t u(\gamma_{n,n}(s))\, ds dt \\ &= \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(c^*(s))\, ds dt \\ &= J(x_0;c^*) \end{aligned}$$

Conclusive remark

Remark: by

$$0 \le \gamma_{n,n} \le N(x_0, T) \text{ a.e. in } [0, T] \quad \forall T > 0$$
$$u(\gamma_{n,n}) \rightharpoonup u(c^*) \text{ in } L^1([0, T]) \quad \forall T > 0$$

it follows that the control c^* which is *optimal* at x_0 is actually **locally bounded** in $[0, +\infty)$ (while the *admissible* controls could be unbounded in some compact subset of $[0, +\infty)$).

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There exists a locally bounded optimal control (consumption strategy) for every fixed initial state (capital endowment)

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Optimal controls (and consequently optimal states) are strictly positive.
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Next steps

- In order to achieve our ultimate goal in the analysis of the RS model (deep knowledge of the optimal trajectories):
  - Regularity $\mathscr{C}^1$ of the value function
  - Connection between the value function and the Pontryagin's maximum principle (which is non trivial due to the lack of concavity and the presence of state constraint)
  - Uniqueness of the viscosity solution of HJB.

Thank you for the attention!