# A mean field game approach to economic inequalities across space

Fausto Gozzi

work in progress with Giorgio Fabbri, Davide Fiaschi and Giovanni Zanco

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# Outline

# Motivation: ”microscopic” explanation of the time-space evolution of economic variables

- As in the previous talk of Federico our main motivation is to study the space-time evolution of economic variables such as capital and labour.
- The talk of Federico concern the ”macroscopic” time-space evolution of capital. The same for the recent literature on spatial growth (starting from [Brito, ’04], [Brock-Xepapadeas, ’08], [Boucekkine et al, 09], etc).
- We aim at finding a ”microscopic” explanation of such evolution; i.e. looking at the behavior of the economic agents and how their choices determine the ”macroscopic” time-space evolution.

# Related Literature

- For the ”macroscopic” behavior see the above quoted literature on spatial growth and the one on Neoclassical growth models with spatial spillovers (see e.g. [Ertur-Koch, ’07-’11], [Fiaschi-Lavezzi-Parenti, ’18])
- For the ”microscopic” behavior we recall the New Economic Geography (NEG) literature (starting from [Krugman, 1991], see e.g. [Desmet-Rossi Hansberg, ’10-’12] [Allen-Arkolakis, ’14]). Static models.
- For the ”microscopic” modeling techniques we relate to the recent literature in mean field games (see e.g. [Lions, 15], [Carmona-Delarue, ’18]) that recently has found interesting applications in Economics (see e.g. [Achdou et al, 2018]).

# Outline

# State dynamics of the agents

We consider $N$ agents distributed in space (in some domain $D \subseteq \mathbb{R}^2$).
Each agent is identified by

- her *position* $x_t(i)$
- her *human capital* $h_t(i)$

For $i = 1, \dots, N$ the position evolves as follows:

$$\mathrm{d}x_t(i) = v_t(i)\,\mathrm{d}t \qquad (+\varepsilon\,\mathrm{d}W_t(i))$$

$v_t(i)$ is a control: each agent chooses his velocity with the goal of maximising a certain quantity.

The human capital evolves taking account of the interactions among agents (e.g. spillovers). In particular what matters is the *aggregation*.

$$\mathrm{d}h_t(i) = (sh_t(i))^\alpha \bar{h}_t(i)^\xi \,\mathrm{d}t + \chi(h_t(i))\,\mathrm{d}W_t(i)$$

with $s, \alpha, \xi \in (0,1)$, $\alpha + \xi < 1$. $\chi$ is a given bounded map.
Human capital $h$ is divided into *savings* $sh$ and *final production* $(1-s)h$.
The interaction term here is

$$\bar{h}_t(i) := \frac{\sum_{j=1}^N \eta(|x_t(i) - x_t(j)|)h_t(j)}{\left(\sum_{j=1}^N \eta(|x_t(i) - x_t(j)|)\right)^\lambda} =: F(x_t(i), h_t(i); x_t(\cdot), h_t(\cdot))$$

$\lambda$ quantifies the effect of aggregation.
$\eta$ morally is the indicator function of a ball centered at 0.

# Goal of the agents

Each agent wants to maximise

$$J(x_0(i), h_0(i); v_\cdot(i)) := \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t} u_\sigma\left(c_t(i)\right) dt\right]$$

where

$$c_t(i) := -|v_t(i)|^2 + A(x_t(i))[(1-s)h_t(i)]^{1-\gamma}\bar{h}_t(i)^\gamma \geq 0$$

and

$$u_\sigma(c) = \frac{c^{1-\sigma}-1}{1-\sigma}, \qquad \sigma > 0.$$

Note that the goal of each agent depends on the state of the other agents.

# Solving the problem of the agents: HJB equation

Admissible controls:

$$\mathcal{V}^i := \Big\{ v_\cdot(i) : \mathbb{R}_+ \times \Omega \to \mathbb{R}^2, \text{ predictable and s. t. } v_t(i) \in K \text{ and } c_t(i) \geq 0 \text{ for } t \geq 0 \}$$

HJB equation:

$$\rho V(x,h) = shF(x,h,\hat{x},\hat{h}) D_h V(x,h) + \frac{1}{2}\chi(h)^2 \partial^2_{hh} V(x,h) +$$

$$+\beta |\partial_x V(x,h)|^2 + u_\sigma \left( -\beta^2 |\partial_x V(x,h)|^2 + A(x)[(1-s)h]^{1-\gamma} F(x,h,\hat{x},\hat{h})^\gamma \right)$$

where $\beta := \beta(|\partial_x V|, x, h, \hat{x}, \hat{h})$ solves

$$(2\beta)^{1/\sigma} = -\beta^2 |\partial_x V|^2 + A(x)[(1-s)h]^{1-\gamma} F(x,h,\hat{x},\hat{h})^\gamma$$

Goal 1: prove that, when the distribution of the position/capital of the other agents $(\hat{x}, \hat{h})$ is given, the problem of the agents can be solved finding a unique optimal feedback control

$$v_{t*} = \beta(|\partial_x V(x_t, h_t)|, x_t, h_t, \hat{x}_t, \hat{h}_t)\partial_x V(x_t, h_t)$$

given by the feedback map

$$G(D_x V, x, h, \hat{x}, \hat{h}) = \beta(|\partial_x V(x, h)|, x, h, \hat{x}, \hat{h})\partial_x V(x, h)$$

This is work in progress. Main problem: prove that $D_x V$ exists and is Lipschitz continuous.

# Outline

Consider now the distribution of the agents across space

$$S_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\left(x(i),h(i;N)\right)}$$

.

Goal 2: find an equation for such distribution when $N \to +\infty$ (mean field limit).

$$g(x,h) = (sh)^{\alpha},$$
$$\sigma(x,h) = \chi(h)$$
$$b_1(x,y;h,k) = \eta(|x-y|)k$$
$$b_2(x,y;h,k) = \eta(|x-y|)$$

$$\mathrm{d}\langle S_t^N,\varphi\rangle = \frac{1}{N}\sum_{j=1}^{N}\partial_x\varphi(x_t(j),h_t(j))G\,\mathrm{d}t$$

$$+\frac{1}{N}\sum_{j=1}^{N}\partial_h\varphi(x_t(j),h_t(j))g(x_t(j),h_t(j))\left(\frac{\sum_{k=1}^{N} b_1\big(x_t(j),x_t(k);h_t(j),h_t(k)\big)}{\left(\sum_{k=1}^{N} b_2\big(x_t(j),x_t(k);h_t(j),h_t(k)\big)\right)^{\lambda}}\right)^{\xi}\mathrm{d}t$$

$$+\frac{1}{N}\sum_{j=1}^{N}\partial_h\varphi(x_t(j),h_t(j))\sigma(x_t(j),h_t(j))dW_t(j)+\frac{1}{2N}\sum_{j=1}^{N}\partial_{hh}^2\varphi(x_t(j),h_t(j))\sigma(x_t(j),h_t(j))^2\,\mathrm{d}t$$

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$$\mathrm{d}\langle S_t^N,\varphi\rangle = \frac{1}{N}\sum_{j=1}^N \partial_x\varphi(x_t(j),h_t(j))G\,\mathrm{d}t$$

$$+\frac{1}{N}\sum_{j=1}^N \partial_h\varphi(x_t(j),h_t(j))g(x_t(j),h_t(j))N^{\xi(1-\lambda)}\left(\frac{\frac{1}{N}\sum_{k=1}^N b_1\big(x_t(j),x_t(k);h_t(j),h_t(k)\big)}{\Big(\frac{1}{N}\sum_{k=1}^N b_2\big(x_t(j),x_t(k);h_t(j),h_t(k)\big)\Big)^{\lambda}}\right)^{\xi}\mathrm{d}t$$

$$+\frac{1}{N}\sum_{j=1}^N \partial_h\varphi(x_t(j),h_t(j))\sigma(x_t(j),h_t(j))dW_t(j)+\frac{1}{2N}\sum_{j=1}^N \partial_{hh}^2\varphi(x_t(j),h_t(j))\sigma(x_t(j),h_t(j))^2\,\mathrm{d}t$$

$$g(x,h) = (sh)^{\alpha},$$
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$$d\langle S_t^N, \varphi\rangle = \langle S_t^N, G\partial_x\varphi\rangle dt + N^{\xi(1-\lambda)}\langle S_t^N, g\partial_h\varphi \frac{\langle\!\langle S_t^N, b_1\rangle\!\rangle^{\xi}}{\langle\!\langle S_t^N, b_2\rangle\!\rangle^{\lambda\xi}} dt$$

$$+\frac{1}{N}\sum_{j=1}^{N} \partial_h\varphi\left(x_t(j), h_t(j)\right)\sigma\left(x_t(j), h_t(j)\right) dW_t(j) + \frac{1}{2}\langle S_t^N, \sigma^2\partial_{hh}^2\varphi\rangle dt$$

# Limit F-P equation: work in progress

1. if $\lambda = 1$ one gets, formally integrating by parts, the Fokker-Planck PDE

$$\partial_t \mu_t = \frac{1}{2}\partial^2_{hh}\left(\sigma^2 \mu_t\right) - \partial_x \left(G\mu_t\right) - \partial_h \left(g \frac{\langle\!\langle \mu_t, b_1 \rangle\!\rangle^{\xi}}{\langle\!\langle \mu_t, b_2 \rangle\!\rangle^{\lambda\xi}} \mu_t\right) \;;$$

2. if $\lambda > 1$ the interaction term "should" disappear in the limit as $N \to \infty$, giving the "diffusive" behaviour described by the PDE

$$\partial_t \mu_t = \frac{1}{2}\partial^2_{hh}\left(\sigma^2 \mu_t\right) - \partial_x \left(G\mu_t\right) \;;$$

3. if $\lambda < 1$ one does not get a limit from this computation ("big bang"?).

# "Safe" assumptions

- $\lambda = 1$;
- $\sigma$ bounded continuous s.t. $\sigma^2(x) \geq Cx$ and $\sigma \in L^4_{\text{loc}}$ (but need $+\varepsilon\,\mathrm{d}W$ on $x$)

  OR

  $\sigma$ linear and something locally Lipschitz in place of $h^\alpha$ (with the same growth);
- $$\bar{h}_t(i) = \frac{\sum_{j=1}^{N} \eta(|x_t(i;N) - x_t(j;N)|) f(h_t(j;N))}{\sum_{j=1}^{N} \eta(|x_t(i;N) - x_t(j;N)|)}$$

  with $f$ bounded.

Then:

√ tightness;

√ $\sim$ continuity of $\Phi$;

√ existence.

# Outline

HJB depends o the solution of FP and viceversa. Hence:
Goal 3: Prove that, a solution of the coupled system HJB-FP exists (possibly unique).
Then, at least in special cases, find some insights on the solutions and perform some numerics

Thank you for your attention