Optimal Stock–Enhancement Activities for a Spatially Distributed Renewable Resource

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Introduction

Bioeconomic Modelling

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- Standard problem: harvesting
- Here: complimentary problem of stock–enhancement, e. g. cultivation, breeding, feeding or nourishing

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Features of our model:

- spatio-temporal model
- with a one-dimensional (1D) space
- evolution of the resource: growth plus dispersal (diffusion)
- spatially distributed stock–enhancement activity (effort)
- $\rightarrow$ intertemporal optimization problem with distributed control

Background and effective mechanisms

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Background and effective mechanisms

Spatial reaction–diffusion models:

- A reaction–diffusion system may exhibit diffusion–driven (or Turing) instability.
- Similar spatial patterns may also emerge as steady states in diffusive dynamic optimization models;
- this type of mechanism is called *optimal diffusion–induced instability*.
- Patterned optimal steady states (POSS) may result, even though in the absence of diffusion the optimal steady state is necessarily homogeneous.

Our contribution

In this paper:

- Characterisation of the optimal stock–enhancement policies: canonical system (CS)
- Computation of canonical steady states (CSS)
- Calculation of optimal steady states and their associated optimal paths.

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In this paper:

- Characterisation of the optimal stock–enhancement policies: canonical system (CS)
- Computation of canonical steady states (CSS)
- Calculation of optimal steady states and their associated optimal paths.
- POSS may also result in stock–enhancement models!
- More generally: Spatial modelling does make a difference!

A Model of Stock–Enhancement

The agent

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The resource is populated over a one-dimensional area $\Omega \subset \mathbb{R}$.

The agent may enhance the living conditions of the resource by some costly nourishing, feeding or fertilising activity—for short, a stock–enhancing activity—at any location $z \in \Omega$.

Formally, the agent selects a *function* $u(\cdot, t)$ on Ω,

Notation

Let

- $\mathcal{T} \equiv [0, \infty)$: time horizon,
- $x = x(z, t)$: biomass at location $z \in \Omega$ at time $t \in \mathcal{T}$,
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- $u = u(z, t)$: stock–enhancing activity of the agent at time t,
- $J_c(x, u) \equiv \log(x) - \frac{\gamma}{2} u^2$: instantaneous (pointwise) utility,
- J_{ca}: spatial average of J_c over Ω:

$$J_{ca}(t) \equiv \frac{1}{|\Omega|} \int_\Omega J_c\left(x(z,t), u(z,t)\right) \mathrm{d}z.$$

Growth of the biomass

The change of the stock is governed by (i) the net growth of the biomass

$$g(x,u) \equiv ux\left(1-\frac{x}{K}\right)-\delta x,$$

and by (ii) the movement of the resource modelled as diffusion

$$D\Delta x(z,t),$$

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Thus, we have the system dynamics:

$$
\begin{aligned}
\partial_t x(z,t) &= g\left(x(z,t),u(z,t)\right)+D\Delta x(z,t), && \forall z\in\Omega, && (1a)\\
\partial_{\mathbf{n}} x(z,t) &= 0, && \forall z\in\partial\Omega, && (1b)\\
x(z,0) &= x_0(z), && \forall z\in\Omega, && (1c)
\end{aligned}
$$

where $\partial_{\mathbf{n}}$ denotes the outward normal derivative.

Profit maximization

Then, the agent seeks to maximise the average discounted utility

$$\max_{u \in \mathcal{U}} J(x(\cdot), u(\cdot)) \quad \text{where} \quad J(x,u) \equiv \int_0^\infty \mathrm{e}^{-\rho t} J_{ca}(x,u)\,\mathrm{d}t \qquad (2)$$

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subject to (1).

We thus have an *infinite time optimal control problem (OCP) with PDE constraints and a distributed control.*

The formal Hamiltonian for (1) is given by

$$H(x, u, \lambda) = J_c(x, u) + \int_0^\infty \mathrm{e}^{-\rho t} \langle \lambda(\cdot, t), f(x(\cdot, t), u(\cdot, t)) \rangle \,\mathrm{d}t, \quad (3)$$

where $f(x, u) \equiv g(x, u) + D\Delta x$.

Canonical System and Its Steady States

Canonical system (CS)

Necessary optimality conditions for the OCP (2).
Applying Pontryagin's maximum principle yields the CS

$$\partial_t x = ux\left(1-\frac{x}{K}\right)-\delta x+D\Delta x, \quad x(0)=x_0, \tag{4a}$$

$$\partial_t \lambda = \rho\lambda-\frac{1}{x}-\lambda\left(u\left(1-2\frac{x}{K}\right)-\delta\right)-D\Delta\lambda, \tag{4b}$$

with $u=\frac{\lambda}{\gamma}x\left(1-\frac{x}{K}\right)$,

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with $u=\frac{\lambda}{\gamma}x\left(1-\frac{x}{K}\right)$, and Neumann BCs for x and λ:

$$\partial_{\mathbf{n}} x(z,t)=0, \quad \partial_{\mathbf{n}}\lambda(z,t)=0, \quad \forall t\in\mathcal{T}, z\in\partial\Omega. \tag{4c}$$

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along with a transversality condition for λ:

$$\lim_{t\to\infty} \mathrm{e}^{-\rho t} \int_\Omega \lambda(z,t)x(z,t)\mathrm{d}z = 0. \tag{4d}$$

Canonical steady states (CSSs)

We want to solve the CS (4) for u on the infinite time horizon.

A a steady state $(\hat{x}, \hat{\lambda})$ of the CS (4) is called *canonical steady state (CSS)*.

However, solutions of (4) are in general not unique, so that multiple CSS may result.

Canonical steady states (CSS)

Specifically, CSS are the solutions of

$$0 = ux\left(1 - \frac{x}{K}\right) - \delta x + D\Delta x,$$
$$0 = \rho\lambda - \frac{1}{x} - \lambda\left(u\left(1 - 2\frac{x}{K}\right) - \delta\right) - D\Delta\lambda,$$

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associated with (4).

We thus want to

- find multiple CSSs,
- identify *optimal CSSs* with saddle-point property (SPP),
- calculated their associated *optimal paths*,
- compare their values.

Bifurcation Analysis

In case of multiple CSS of (4) the important questions again are:

- which of the CSSs are optimal,
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As an example, we use the parameter specification:

$$D = 0.25,\ K = 1,\ \rho = 0.025,\ \gamma = 10,\ \Omega = (-1.5, 1.5). \qquad (5)$$

And then we let the mortality rate δ vary in the interval $[0.3, 1.2]$ as a bifurcation parameter.

Bifurcation diagrams (1)

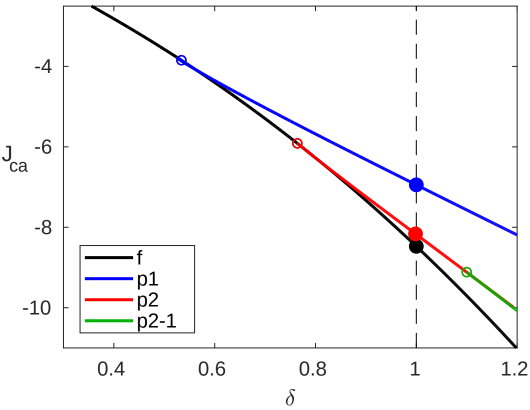

Figure 1: Bifurcation diagram for J_{ca}. Bifurcations at $\delta_1 = 0.5340$ and $\delta_2 = 0.7641$ on the branch `f` of flat CSSs. The branches `p1` and `p2` feature *patterned* CSSs; and solutions on `p1` have the SPP.

Bifurcation diagrams (2)

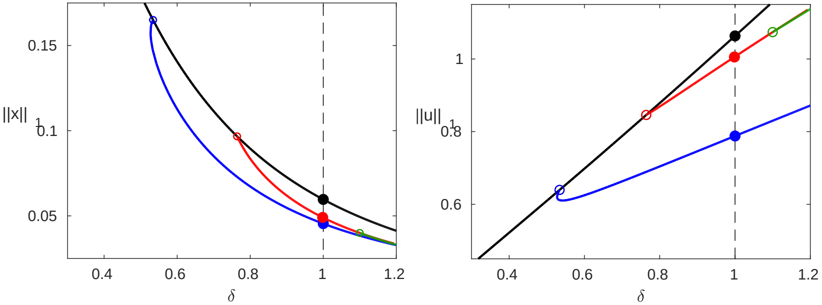

Figure 2: Corresponding bifurcation diagrams for the state variable and the control.

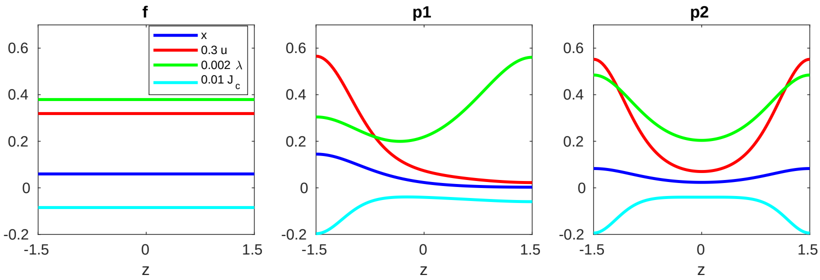

Figure 3: CSSs for $\delta = 1$: a flat (f, left) and two patterned (p1, middle; p2, right) CSS, with state variable, control, co-state and yield.

Canonical Paths

Definition

Given an initial state x_0, we are interested in time dependent solutions $(x(z,t), \lambda(z,t))$ of (4) connecting x_0 to some CSS $(\hat{x}, \hat{\lambda})$:

$$x(z,0) = x_0(z) \quad \text{and} \quad \lim_{t\to\infty} (x(z,t), \lambda(z,t)) = (\hat{x}(z), \hat{\lambda}(z)) \quad \forall z.$$

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Such a path is called *canonical path* (CP).

Conversely, we can consider a given CSS $(\hat{x}, \hat{\lambda})$ and ask from which x_0 it can be reached by a suitably chosen CP.

The CPs show how to govern the system to the optimal CSS in an optimal way.

Canonical path from the flat to the patterned CSS (1)

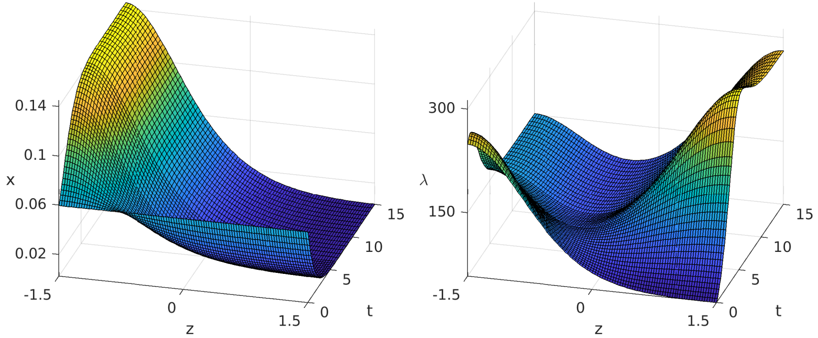

Figure 4: CP from the flat CSS `f` to the patterned CSS `p1`: state and costate.

Canonical path from the flat to the patterned CSS (2)

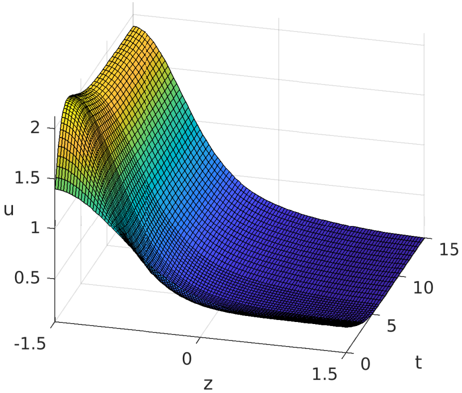

Figure 5: CP from the flat CSS `f` to the patterned CSS `p1`: control.

Objective values of the CSSs and their optimality

CSS	J–value	SPP	Remarks
FCSS f	-338.95	no	not optimal, dominated by the CP to p1 with value $J = -277.77$
PCSS p2	-327.16	no	not optimal, dominated by the CP to p1 with value $J = -279.82$
PCSS p1	-277.80	yes	optimal, hence a POSS, with a large (probably global) domain of attraction

Table 1: Properties of the canonical steady states for problem (1), for parameter setting (5) and $\delta = 1$.

Conclusion

Results

- Stock–enhancement problem for a spatially distributed renewable resource.
- OCP with PDE constraints and a distributed control.
- CS and associated steady states (CSS).
- Bifurcation analysis helped to detect multiple CSS.

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- Stock–enhancement problem for a spatially distributed renewable resource.
- OCP with PDE constraints and a distributed control.
- CS and associated steady states (CSS).
- Bifurcation analysis helped to detect multiple CSS.
- We explored the optimal CPs.
- We showed that patterned optimal steady states result!
- Hence, the intuitive policy proposal calling for an equal distribution of the efforts may be totally misled.

Thank you for your attention!