Growth and agglomeration in the heterogeneous space: a generalized AK approach

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1. Review of the classical AK model

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3. Research developments
In the classical AK model in continuous time, everything happens in the same place: the economic world is a single point.

⇒ Space dimension = 0.

The production function is linear in capital, i.e. the output $y$ is proportional to the capital level $k$:

$$y = \tilde{A} k,$$

where $\tilde{A}$ is a parameter representing the technological level.

Letting $c(t) \geq 0$ be the consumption rate at time $t \in \mathbb{R}_+$, the dynamics of capital $k(t)$ is described by the ODE

$$\begin{cases}
    k'(t) = y(t) - \delta k(t) - c(t) = (\tilde{A} - \delta) k(t) - c(t) = A k(t) - c(t), \\
    k(0) = k_0,
\end{cases}$$

where

- $k_0 > 0$ is the initial capital;
- $\delta \geq 0$ is the depreciation rate of capital;
- $A := \tilde{A} - \delta$. 


In this framework, one poses the problem of maximizing the discounted intertemporal utility from consumption represented by the functional

\[ \int_0^\infty e^{-\rho t} u(c(t)) \, dt, \]

where \( \rho > 0 \) is a discount factor and \( u : \mathbb{R}_+ \to \mathbb{R} \) is a utility function, over the set of admissible consumption strategies

\[ \mathcal{A}(k_0) = \{ c \in L^1_{loc}(\mathbb{R}_+; \mathbb{R}_+) : k(t; k_0, c(\cdot)) > 0 \ \forall t \in \mathbb{R}_+ \} \]

where

\[ L^1_{loc}(\mathbb{R}_+; \mathbb{R}_+) = \left\{ c : \mathbb{R}_+ \to \mathbb{R}_+ : \int_0^T |c(t)| \, dt < \infty \ \forall T > 0 \right\}. \]
Review of the classical AK model

Method of solution: Dynamic Programming

- One defines the *value function*, the optimum of the problem for generic initial datum:

  \[ V(k_0) = \sup_{c(\cdot) \in A(k_0)} \int_0^\infty e^{-\rho t} u(c(t)) \, dt. \]

- One associates to this function an ODE, the Hamilton-Jacobi-Bellman equation: in this case it is

  \[ \rho \nu(k_0) = Ak_0 + u^*(\nu'(k_0)), \]

  where

  \[ u^*(p) = \sup_{c \in \mathbb{R}_+} \{ u(c) - cp \}. \]

- One solves the above ODE.

- Given a solution \( \nu \) of the ODE one proves a *verification theorem* showing that

  \[ \nu(k_0) = V(k_0) \]

  and, as a byproduct, constructing an optimal consumption strategy in feedback form:

  \[ c^*(t) = \arg\max_{c \in \mathbb{R}_+} \{ u(c) - cv'(k(t; k_0, c^*(t))) \}. \]
In the case
\[ u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad \sigma \in (0, 1) \cup (1, +\infty), \]
this program can be carried out in a nice explicit form: under the assumption
\[ \rho > A(1 - \sigma). \]

- The value function is
\[ V(k_0) = v(k_0) = \frac{1}{1-\sigma} \left[ \frac{\rho - A(1 - \sigma)}{\sigma} \right]^{-\sigma} k_0^{1-\sigma}. \]

- The optimal consumption path and the optimal capital path are
\[ c^*(t) = \frac{\rho - A(1 - \sigma)}{\sigma} k_0 e^{\frac{A-\rho}{\sigma} t}, \quad k^*(t) = k_0 e^{\frac{A-\rho}{\sigma} t}. \]

- The growth rate of the economy is
\[ g := \frac{A - \rho}{\sigma}, \]
which can be positive or negative according to \( \rho < A \) and \( \rho > A \).
1. Review of the classical AK model

2. An AK spatial model with non-homogeneous data

3. Research developments
A series of papers in the last decade have tried to merge the spatial dimension with growth theory models, using a “diffusion hypothesis”.

Brito (2004) is first one formulating a Ramsey model with continuous space.

Boucekkine, Camacho and Fabbri (JET, 2013) are the first ones who completely solve a spatial endogenous growth model. They study an AK economy on a circle and they prove that, under certain condition, the detrended capital across locations converges to a constant.

In our paper we generalize the model and the results of BCF (2013) — our benchmark.
Model

- The (geographical) space is modeled as the unit circle of the plane:

\[ T := \left\{ x \in \mathbb{R}^2 : |x| = 1 \right\}. \]

**Note 1:** \( T \) can be simply seen as the interval \( \theta \in [0, 2\pi] \) with the identification of the two extrema.

**Note 2:** The choice of the unit circle allows to avoid the specification of boundary conditions.

- \( x \in T \) represents location (e.g., \( x = \text{Roma, Vienna, etc.} \)).
For $t \in \mathbb{R}_+, x \in \mathbb{T}$, consider the following variables:

- $k(t, x) = \text{capital at } (t, x)$;
- $c(t, x) = \text{per capita consumption rate at } (t, x)$;
- $A(x) := \tilde{A}(x) - \delta(x) = \text{tech. level minus depreciation rate at } x$;
- $N(x) = \text{population density at } x$.

The capital accumulation law at $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ is

$$\frac{\partial k}{\partial t}(t, x) = A(x)k(t, x) + \tau(t, x) - N(x)c(t, x)$$

where $\tau(t, x)$ is the household's net trade balance at $(t, x)$.
Brito’s modelisation of $\tau$.
Given the region $X = [x, x + h]$, the net trade balance is

$$\int_x^{x+h} \tau(t, \xi) \, d\xi.$$ 

Imposing it equal to the capital flow across the boundaries $x$ and $x + h$ and assuming that the latter is the opposite of the derivative of the capital at the boundary (Fick’s law):

$$\int_x^{x+h} \tau(t, \xi) \, d\xi = \frac{\partial k(t, x + h)}{\partial x} - \frac{\partial k(t, x)}{\partial x},$$

Considering the limit $h \to 0^+$ we get

$$\tau(t, x) = \frac{\partial^2 k}{\partial x^2}(t, x).$$
Using this fact in the original state equation we get the evolution of capital

\[
\begin{align*}
\frac{\partial k}{\partial t} (t, x) &= \frac{\partial k}{\partial x^2} (t, x) + A(x) k(t, x) - N(x) c(t, x), \\
k(0, x) &= k_0(x).
\end{align*}
\]

The planner’s problem is

\[
\text{Max } \int_0^\infty e^{-\rho t} \left( \int_T N(x) \frac{(c(t, x))^{1-\sigma}}{1 - \sigma} \, dx \right) \, dt,
\]

where \(\sigma \in (0, 1) \cup (1, +\infty)\), over the set of admissible consumptions policies

\[
\mathcal{A}(k_0) = \{ c : L^1_{loc} (\mathbb{R}_+ \times T; \mathbb{R}_+) : k^{k_0, c}(t, x) > 0 \ \forall(t, x) \in T \times \mathbb{R}_+ \}.
\]
An AK spatial model with non-homogeneous data

Contribution

We extend the model of BCF (2013) in two directions:

(i) We allow for a \textit{space-heterogeneous distribution of the technology parameter} among locations; i.e. $A = A(x)$.

(ii) We allow for any time-independent specification of the \textit{heterogeneous spatial distribution of the agents}; i.e. we consider a population density $N = N(x)$.

(In BCF the only non-homogeneous datum is the initial distribution $k_0$.)

We derive the \textit{long-run profile} of the detrended variables. In particular:

(i) We show the detrended variables converge towards a spatial non-uniform capital distribution (unlike in BCF) that we characterize;

(ii) We describe the effects of technological and population discrepancy on the shape the long-run configuration.
Methodology

- We rephrase the control system in an abstract setting.

- The PDE is seen as an ODE in an infinite-dimensional space — with uncontrolled dynamics described by the differential operator

\[ \mathcal{L}u := \left[ \frac{\partial^2}{\partial x^2} + A(\cdot) \right] u. \]

- Dynamic Programming can be applied — with some care — in infinite dimension.

- Explicit solutions are possible!
Solution and results

- There is a countable discrete set of eigenvalues \( \{\lambda_n\}_{n \geq 0} \subset \mathbb{R} \), i.e. such that the ODE on \( \mathbb{T} \)
  \[ \mathcal{L}u = \lambda_n u \]
  has a nonvanishing solution.
- The above set can be ordered in decreasing way.
- There exists an orthonormal basis of \( L^2(\mathbb{T}) \) of eigenfunctions \( \{e_n\}_{n \geq 0} \) corresponding to the sequence of eigenvalues \( \{\lambda_n\}_{n \geq 0} \).
- The highest eigenvalue, \( \lambda_0 \), is associated to a unique eigenfunction and this is the only eigenfunction without zeros.
- We denote by \( e_0 \) the unique normalized positive eigenfunction corresponding to \( \lambda_0 \).
Theorem (Optimal consumption and aggregate capital)

Assume that

\[ \rho > \lambda_0 (1 - \sigma). \]

Let

\[ K(t) := \int_T k(t, x) \, dx \]

be the aggregate capital at time \( t \). Define

\[ \alpha_0 := \left( \frac{\sigma}{\rho - \lambda_0 (1 - \sigma)} \int_T e_0(x) \left( -\frac{(1 - \sigma)}{\sigma} N(x) \right) \right)^{\frac{\sigma}{1 - \sigma}}, \]

\[ c^*(t, x) := \left( \int_T \alpha_0 e_0(\xi) K_0(\xi) \, d\xi \right) \left( \alpha_0 e_0(x) \right)^{-1/\sigma} e^{gt}, \]

where

\[ g := \frac{\lambda_0 - \rho}{\sigma}. \]

If the corresponding capital trajectory \( k^*(t, x) \) remains positive, \( c^*(\cdot, \cdot) \) is an optimal consumption policy and the corresponding optimal aggregate capital at time \( t \) is

\[ K^*(t) = K(0) e^{gt}. \]
Comments

- The expression of the optimal consumption contains:
  - a space-independent coefficient
    \[
    \left( \int T \alpha_0 e_0(\xi) K_0(\xi) d\xi \right);
    \]
  - a space-dependent coefficient
    \[
    (\alpha_0 e_0(x))^{-1/\sigma}.
    \]
- \(c^*(t, \cdot)\) is not uniform in space, as \(e_0\) is not, unless \(A(\cdot)\) is constant (the latter is the case of BCF).
- The space-shape of \(c(t, x)\) depends on \(A(\cdot)\) only via \(e_0\).
- The level of \(c(t, x)\) depends both on \(A(\cdot)\) and \(N(\cdot)\) via \(\alpha_0\).
Theorem (Long-run distribution of capital)

Assume
\[ \lambda_0 (1 - \sigma) < \rho < \lambda_0 - \lambda_1 \sigma. \]

Define the detrended optimal path
\[ k_g^*(t, x) := e^{-gt} k^*(t, x), \quad t \geq 0. \]

Then
\[ k_g^*(t, x) \xrightarrow{t \to \infty} \left( \int_T \alpha_0 k_0(\xi) e_0(\xi) \, d\xi \right) \left( \frac{e_0(x)}{\alpha_0} + \sum_{n \geq 1} \frac{\beta_n}{\lambda_n - g} e_n(x) \right) \]

where, for \( n \geq 1 \),
\[ \beta_n := \int_T (\alpha_0 e_0(\xi))^{-1/\sigma} N(\xi) e_n(\xi) \, d\eta. \]
Comments

In the case of constant $A, N$, we have

$$\lambda_0 = A, \quad \lambda_1 = (A - 1).$$

We retrieve the same conditions (and results) of BCF (2013).

Unlike in BCF (2013), here we do not have convergence to a constant function.

Indeed the detrended capital here converges to an articulated expression, depending on the whole technological and population distributions.
A few more words on the mathematical side

(i) We rewrite the problem in the Hilbert space $L^2(\mathbb{T})$:

$$
\begin{aligned}
\begin{cases}
K'(t) = \mathcal{L}K(t) - c(t)N, & t \in \mathbb{R}^+ , \\
K(0) = K_0 = k_0(\cdot),
\end{cases}
\end{aligned}
$$

where

$$
K : \mathbb{R}^+ \to L^2(\mathbb{T}), \quad c : \mathbb{R}^+ \to L^2_+(\mathbb{T}),
$$

with the formal equalities

$$
[K(t)](x) = K(t, x), \quad [c(t)N](x) = c(t, x)N(x),
$$

and target and

$$
\max \int_0^{\infty} e^{-\rho t} \mathcal{U}(c(t)) dt,
$$

where

$$
\mathcal{U}(q) = \int_{\mathbb{T}} \frac{q(x)^{1-\sigma}}{1-\sigma} dx, \quad q \in L^2_+(\mathbb{T}).
$$
(ii) We introduce the (infinite dimensional) Hamilton-Jacobi-Bellman (HJB) equation

$$\rho \nu(K) = \langle K, L \nabla \nu(K) \rangle + \sup_{c \in L^2_+} \{ \mathcal{U}(c) - \langle cN, \nabla \nu(K) \rangle \},$$

(iii) We find a solution $\nu$ to HJB and prove by verification that it coincide with the value function $V : L^2(\mathbb{T}) \rightarrow \mathbb{R}$:

$$V(K) = \nu(K) = \frac{\langle K, \alpha_0 e_0 \rangle^{1-\sigma}}{1-\sigma},$$

where

$$\alpha_0 := \left( \frac{\sigma}{\rho - \lambda_0 (1-\sigma)} \int_{\mathbb{T}} e_0(x) \left( -\frac{1-\sigma}{\sigma} \right) N(x) \, dx \right)^{\frac{\sigma}{1-\sigma}},$$

(iv) We use the value function to write the optimal current value of the control as function of the value of the current state (optimal feedback, again from $L^2(\mathbb{T})$ to $\mathbb{R}$).

(v) We use the feedback relation to find optimal trajectories of the state and of the control.
Numerics

We show now a series of numerical simulations. We take:

- $\rho = 3\%$ (consistent e.g. with the data of Lopez, 2008)
- $\sigma = 5$ (coherent with the results found e.g. by Barsky et al., 1997)
- we use the following technological distribution $A(\cdot)$ on $\mathbb{T} \simeq [0, 2\pi]$

It has a pick at the point $\pi$ (the “core”) and attains lower values in the further locations (the “periphery”).
An AK spatial model with non-homogeneous data

The technology space discrepancy effect

![Graphs showing the limit distribution of detrended variables](image)
Comments on the effects of technological discrepancy

- Non-uniform spatial technological distribution, the population is constant with density everywhere equal to 1.
- Capital tends to accumulate at the core where it is more productive while areas with smaller technological endowment remain behind
- Higher productivity of the capital in the core locations pushes the planner to increase investments and thus savings relatively more in these regions, as a byproduct the planner privileges consumption in peripheral regions
- The capital distribution is much less concentrated than the technological level: we have indeed an endogenous spatial spillover effect that is the combined result both of the capital exogenous diffusivity and the endogenous investment and consumption decisions by the planner.
An AK spatial model with non-homogeneous data

Core-periphery and population effect at work
Comments on interactions between population and technological distribution

- We consider a concentration of capital productivity and population density in the same areas (a quite frequent configuration).

- Two distinct motivations drive the planner: on the one hand, she will tend to invest more in the more productive areas; on the other hand, she is tempted to assign a reasonable per capita level of consumption in each region.

- The aggregate investment in more productive areas remains relatively higher, but the effect is mitigated because aggregate consumption is higher in these areas as well.

- The distribution of the long-run detrended aggregate capital is much more uniform in the case of non-uniformly distributed population, so that capital accumulates relatively more in less productive areas.

- The change in the population distribution translates in a sort of loss of efficiency of the system: the per-capita consumption in the new configuration is always smaller than in the original one at any location.
1. Review of the classical $AK$ model

2. An $AK$ spatial model with non-homogeneous data

3. Research developments
Pb 1: A geographical model with capital and pollution

WP with the same co-authors

For $t \in \mathbb{R}_+$, $x \in \mathbb{T}$, consider the following variables:

- $k(t, x) =$ capital at $(t, x)$;
- $p(t, x) =$ pollution at $(t, x)$;
- $c(t, x) =$ consumption rate at $(t, x)$;
- $\tilde{A}(x) =$ technological level at $x$;
- $\delta(x) =$ capital depreciation rate at $x$;
- $w(x) =$ ecological awareness at $x$;
- $r(x) =$ natural regeneration rate at $x$;
- $\alpha(x) =$ coefficient of pollution production at $x$. 
• Evolution of state variables $k(t, x), p(t, x)$.

• Evolution of capital at $(t, x) \in \mathbb{R}_+ \times \mathbb{T}$:

\[
\frac{\partial k}{\partial t}(t, x) = (\tilde{A}(x) - \delta(x))k(t, x) - c(t, x);
\]

→ Note that there is no diffusion of capital: this is actually a family of ODEs parametrized in $x$.

• Evolution of pollution at $(t, x) \in \mathbb{R}_+ \times \mathbb{T}$:

\[
\frac{\partial p}{\partial t}(t, x) = \underbrace{\frac{\partial^2 p}{\partial x^2}(t, x)}_{\text{diffusion of poll.}} - \underbrace{r(x)p(t, x)}_{\text{decay of poll.}} + \underbrace{\alpha(x)\tilde{A}(x)k(t, x)}_{\text{production of poll.}};
\]
The planner’s problem is

\[
\text{Max } \int_0^\infty e^{-\rho t} \left( \int_T \left( \frac{c(t, x)^{1-\sigma}}{1 - \sigma} - w(x)p(t, x) \right) dx \right) dt,
\]

where \( \rho > 0, \sigma \in (0, 1) \cup (1, +\infty) \), over the consumption policies \( c(t, x) \) keeping the capital positive.

**Goal:** obtain explicit solutions and provide a numerical analysis on the spirit of the one described above.

**Extension:**
We are also trying to set and study a game version of this problem.
Consider an economy with one representative good.

Given

\[ t \in \mathbb{R}_+ = \text{set of times} \]
\[ x \in \mathbb{T} = \text{geographical space}, \]
\[ \omega \in \Omega = \text{probability space}, \]

consider the following variables:

- \( \Phi(t, x, \omega) = \) price of the good at time \( t \), location \( x \), and “state of nature” \( \omega \);
- \( c(t, x, \omega) = \) cost per unit of investment at time \( t \), location \( x \), and “state of nature” \( \omega \);
- \( K(t, x, \omega) = \) capital at time \( t \), location \( x \), and “state of nature” \( \omega \);
- \( I(t, x, \omega) = \text{cumulative investment up to time } t, \text{ at location } x, \text{ and in the “state of nature” } \omega: \text{ it is a nondecreasing process for each } x \) (irreversible investment).
The equation stating the evolution of capital is the stochastic controlled PDE
\[
dK(t, x, \omega) = [\mathcal{A}K(t, \cdot, \omega)](x)dt + dl(t, x, \omega),
\]
where
- \( \mathcal{A} : D(\mathcal{A}) \subseteq L^2(\mathbb{T}) \to L^2(\mathbb{T}) \) is a linear operator accounting for the spatial effects, e.g.
  \[
  \mathcal{A} = \frac{\partial^2}{\partial x^2} - \delta(\cdot),
  \]
  where \( \delta(x) > 0 \) is the depreciation rate of capital at location \( x \);
- \( l \) is the control process lying in a suitable set of admissible control processes \( \mathcal{I} \).
The planner’s problem is \( \max_{I \in \mathcal{I}} \) the functional

\[
\mathbb{E} \left[ \int_0^T e^{-\rho t} \left[ \left( \int_{\mathcal{T}} F(\Phi(t, x, \cdot), K(t, x, \cdot)) dx \right) dt - \int_{\mathcal{T}} c(t, x, \cdot) dl(t, x, \cdot) dx \right] \right]
\]

where \( \rho > 0 \) and \( F : \mathbb{R}^2_+ \to \mathbb{R}_+ \) is a revenue function.

**Goal:** Characterize the optimal policies by first order conditions and provide examples with explicit solutions.
THANKS FOR THE ATTENTION