Adaptive Network Dynamics

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www.multiscale.systems
Research Area(s)

- Multiscale Methods
- Stochastic Systems

- Nonlinear Dynamical Systems

- Nonlocality & Patterns
- Network Dynamics
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Adaptive Networks: Setting and Notation

- vertices/nodes $\mathcal{V}$, edges/links $\mathcal{E}$, graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$;
- (weighted) adjacency matrix $A = (a_{ij}) \in \mathbb{R}^{N \times N}$, $i, j \in \{1, 2, \ldots, |\mathcal{V}| =: N\}$;
- vertex state $u_i$; e.g. $u_i \in \{\bullet, \bullet\}$.

starting graph / initial condition

$t = 0$
Adaptive Networks: Setting and Notation

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node dynamics $\leftrightarrow$ link dynamics

$t=1$
Adaptive Networks: Setting and Notation

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node dynamics $\leftrightarrow$ link dynamics

$t=2$
Example 1: The Adaptive SIS Model (Gross et al)

- **Infected**
- **Susceptible**

\[
P \quad 1 - P
\]

\[
1 - r
\]

\[
1 - w
\]

\[
\text{infection}
\]

\[
\text{recovery}
\]

\[
\text{re-wiring}
\]
Example 1: The Adaptive SIS Model - New Results

Use pair-approximation \( l_{abc} = \frac{l_{ab}l_{bc}}{b} \) + noise ⇒

\[
i' = \lambda \left( \frac{\mu}{2} - l_{II} - l_{SS} \right) - ri + \text{noise}, \\
(l_{II})' = p \left( \frac{\mu}{2} - l_{II} - l_{SS} \right) \left( \frac{\frac{\mu}{2} - l_{II} - l_{SS}}{1 - i} + 1 \right) - 2rl_{II} + \text{noise}, \\
(l_{SS})' = (r + w) \left( \frac{\mu}{2} - l_{II} - l_{SS} \right) - 2\lambda l_{SS} \frac{\frac{\mu}{2} - l_{II} - l_{SS}}{1 - i} + \text{noise}.
\]

epidemic → healthy

healthy → epidemic

Example 1: The Adaptive SIS Model - New Results*

- epidemic → healthy upon decreasing $\lambda$
- healthy → epidemic upon increasing $\lambda$

- early-warning signs for critical transitions
- link density $\text{Var}(l_{SS}(\lambda)) = \mathcal{O} \left( (\lambda - \lambda_{tc})^{-1} \right)$ as $\lambda \to \lambda_{tc}$.

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Example 1: The Adaptive SIS Model - New Results

Idea: utilize network measures as warning signs

Effective branching ratio $\kappa := \frac{[SSI]}{[SI]}$

$\Rightarrow$ not a warning sign, despite scaling!

Example 1: The Adaptive SIS Model - New Results†

Idea: utilize network measures as warning signs

Clustering coefficient

\[ C := \frac{\text{Tr}(A^3)}{\sum_{ij}(A^2)_{ij} - \text{Tr}(A^2)} \]

⇒ excellent warning sign, local max!

Example 1: The Adaptive SIS Model - New Results

**Idea:** utilize network measures as warning signs

**Summary** (epidemic $\rightarrow$ healthy transition):

- Effective branching ratio $\kappa := \frac{[SSI]}{[SI]}$ ✓
- Clustering coefficient $C := \frac{\text{Tr}(A^3)}{\sum_{ij}(A^2)_{ij} - \text{Tr}(A^2)}$ ✓
- SI link density, triplet densities ✓
- Harmonic mean distance X
- Assortativity ✓
- Variance of degree distribution X
- Eigenvalue gap ✓

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Example 2: “Synaptic” Plasticity (Bornholdt-Rohlf)

1. nodes $v_i(t) \in \{\pm 1\}$, directed edges $a_{ij}(t) \in \{-1, 0, +1\}$.
2. dynamical update rule ($t = 0$, random graph), define

$$f_i(t) = \sum_j a_{ij}(t)v_j(t) + \mu v_i(t) + \sigma r_i, \quad r_i \sim \mathcal{N}(0, 1)$$

$$v_i(t + 1) = \begin{cases} 
\text{sgn}[f_i(t)] & \text{if } f_i(t) \neq 0, \\
v_i(t) & \text{if } f_i(t) = 0.
\end{cases}$$
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\end{cases}$$

3. $T_v$ node dynamics steps, $T_a := \lfloor T_v/2 \rfloor$, measure activity

$$\mathcal{U}_i := \frac{1}{T_v - T_a} \left[ \sum_{t=T_a}^{T_v-1} v_i(t) \right].$$

4. topological update rule, choose one node $i$ randomly

- $|\mathcal{U}_i| > 1 - \delta$ create an edge $a_{ij}(t) \neq 0$,
- $|\mathcal{U}_i| \leq 1 - \delta$ delete an edge $a_{ij}(t) = 0$. 
Example 2: “Synaptic” Plasticity - New Results

- SOC = steady state near (fast subsystem) bifurcation point
- Optimal finite noise $\leftrightarrow$ steady stochastic resonance (SSR)
- Optimal time scale $\leftrightarrow$ time scale resonance (TSR)

Example 3: Adaptive Snowdrift Game (Pacheco et al)

Payoff matrix: \( M = \begin{pmatrix} b - c/2 & b - c \\ b & 0 \end{pmatrix} \)

- \( c = \) cost of cooperation, and \( b = \) benefit;
- strategies: cooperation \( u_i = C \) or defection \( u_i = D \);
- total payoff for node \( i \) is \( \pi_i = \sum_{j: a_{ij}=1} M_{ij} \).
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Dynamics & Adaptivity:

- \( n_u = \) fraction of agents using strategy \( u \), define average payoff
  \[ \psi(u) := \frac{1}{n_uN} \sum_{i: u_i=u} \pi_i. \]
- probability to change dynamics (along \( a_{ij} \) edge)
  \[ f_\beta(i,j) = \left( 1 + e^{-\beta[\psi(u_i)-\psi(u_j)]} \right)^{-1} \]
- re-wire an edge with probability \( f_\alpha(i,j) \).
Example 3: Adaptive Snowdrift Game - New Results

Re-wire \( p \), adopt \( 1 - p \) based upon performance.

---

Example 3: Adaptive Snowdrift Game - New Results

Re-wire ($p$), adopt $(1 - p)$ based upon performance.

- Hopf bifurcation: variance growth warning
  \[ \text{Var}(x_{1,2}(p)) = \mathcal{O} \left( (p - p_{\text{Hopf}})^{-1} \right) \text{ as } p \to p_{\text{Hopf}}. \]

- Periodic-to-homoclinic transition: period blow-up warning
  \[ T^{-1}(p) = \mathcal{O} \left( (p - p_{\text{hom}}) \right) \text{ as } p \to p_{\text{hom}}. \]

- Saddle transition(s): log-distance reduction warning.

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Example 4: Evolution of Early Life (Jain-Krishna)

1. nodes \( v_i(t) \in [0, 1] \), directed edges \( a_{ij}(t) \in \{0, 1\} \).

2. dynamical update rule \((t = 0, \text{random graph})\), catalytic ODEs

\[
v' = A^T v - \|A^T v\|_1 v, \quad \|z\|_1 := \sum_{i=1}^{N} |z_i|
\]
Example 4: Evolution of Early Life (Jain-Krishna)

1. nodes $v_i(t) \in [0, 1]$, directed edges $a_{ij}(t) \in \{0, 1\}$.

2. dynamical update rule ($t = 0$, random graph), catalytic ODEs

$$v' = A^T v - \|A^T v\|_1 v, \quad \|z\|_1 := \sum_{i=1}^{N} |z_i|$$

3. look at $v(T)$, $T$ “big”; define weakest vertex indices

$$\mathcal{J}_* := \{j \in \{1, 2, \ldots, N\} : v_j = \min_{k} v_k\}.$$

4. topological update rule:
   - randomly select $j \in \mathcal{J}_*$,
   - eliminate $v_j$ and all its edges,
   - insert a new vertex, links via Erős-Renyi rule, probability $p \in (0, 1)$. 
Example 4: Evolution of Early Life - New Results

Rigorous results - vertex dynamics:
- solutions generically converge to steady state in \( \{z \in \mathbb{R}^d : \sum_{i=1}^{N} z_i = 1\} \)
- relation to projective space and algebraic characterization
- exponential fast convergence \( \Rightarrow \) “fast dynamics”

Formal calculations - edge dynamics:
- formation time of the first cycle
- formation time of a single autocatalytic set \( (ACS) = \mathcal{G} \)
- \( Np < 1 \) \( \Rightarrow \) “slow dynamics”

\( \Rightarrow \) multiscale / fast-slow system explains spike dynamics

Example 5: Consensus Networks

- graph Laplacian $L := D - A$ where $D_i = \text{diag}(\sum_j a_{ij})$,
- allow weights $\leftrightarrow a_{ij} \in \mathbb{R}$,
- state $v_i(t) \in \mathbb{R}$, consensus $v_i(t) = v_j(t) \; \forall t \geq T$.

Dynamical update rule & topological update rule

$v' = -Lv = (D(t) - A(t))v$,
$A' = \varepsilon g(v, A)$. 
Example 5: Consensus Networks - New Results

- first results: small-scale “triangle” graph
- methods: fast-slow dynamical systems analysis

Example 5: Consensus Networks - New Results

- first results: small-scale “triangle” graph
- methods: fast-slow dynamical systems analysis

⇒ non-trivial dynamics near consensus manifold!

\[ \text{III} \]

Example 6: Simplicial Adaptive Voters (Horstmeyer-K.)
Example 6: Simplicial Adaptive Voters - New Results

- $p$ small $\Rightarrow$ absorption into one-opinion state
- $p$ large $\Rightarrow$ fragmentation into two graphs

$\Rightarrow$ speed-up via peer-pressure (e.g. edge depletion rate)!

Example 6: Simplicial Adaptive Voters - New Results

- $p$ small $\Rightarrow$ absorption into one-opinion state
- $p$ large $\Rightarrow$ fragmentation into two graphs

$\Rightarrow$ speed-up via peer-pressure (e.g. drift velocity)!

The Last Slide...

For further information and links to my papers/books:

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Thank you very much for your attention!!