Adaptive Network Dynamics

Christian Kuehn

Lichtenberg Professor for
Multiscale and Stochastic Dynamics
Technical University of Munich (TUM)

External Faculty, Complexity Science Hub Vienna

`www.multiscale.systems`

VolkswagenStiftung
DFG Deutsche Forschungsgemeinschaft
IAS TUM Institute for Advanced Study
Mathematisches Forschungsinstitut Oberwolfach
SEVENTH FRAMEWORK PROGRAMME
ÖAW Österreichische Akademie der Wissenschaften

Research Area(s)

Multiscale Methods

Stochastic Systems

Nonlinear Dynamical Systems

Nonlocality & Patterns

Network Dynamics

Research Area(s)

Multiscale Methods

- fast-slow systems
- perturbation methods
- geometric desingularization
- complex oscillations
- ...

Stochastic Systems

- path-based methods
- early-warning signs
- stochastic PDEs
- (rigorous) computation
- ...

Nonlocality & Patterns

- fractional & nonlocal PDEs
- numerical continuation
- travelling waves
- bifurcation theory
- ...

Network Dynamics

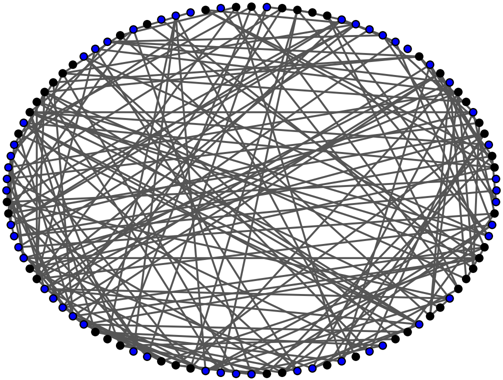
- adaptive networks
- graph limits
- data analysis
- moment closure
- ...

Adaptive Networks: Setting and Notation

- vertices/nodes $\mathcal{V}$, edges/links $\mathcal{E}$, graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$;
- (weighted) adjacency matrix $A = (a_{ij}) \in \mathbb{R}^{N \times N}$, $i, j \in \{1, 2, \ldots, |\mathcal{V}| =: N\}$;
- vertex state u_i; e.g. $u_i \in \{\bullet, \bullet\}$.

starting graph / initial condition

$t = 0$

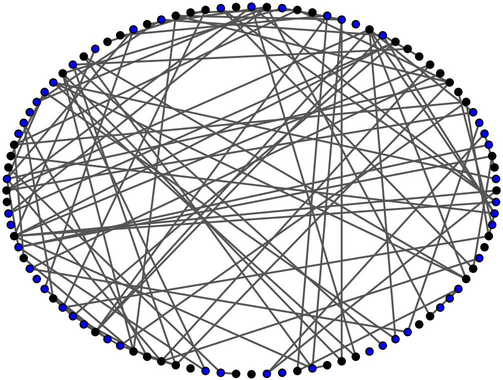


Adaptive Networks: Setting and Notation

- vertices/nodes $\mathcal{V}$, edges/links $\mathcal{E}$, graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$;
- (weighted) adjacency matrix $A = (a_{ij}(t)) \in \mathbb{R}^{N \times N}$, $i, j \in \{1, 2, \ldots, |\mathcal{V}| =: N\}$;
- vertex state $u_i(t)$; e.g. $u_i(t) \in \{\bullet, \bullet\}$.

node dynamics $\leftrightarrow$ link dynamics

t=1

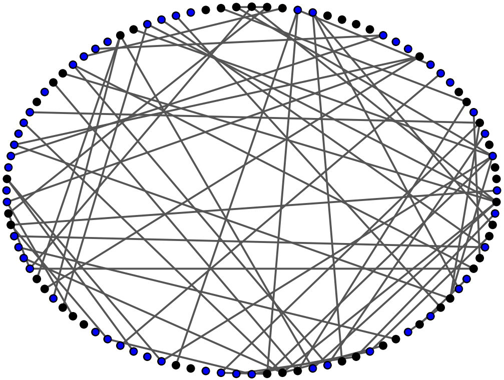


Adaptive Networks: Setting and Notation

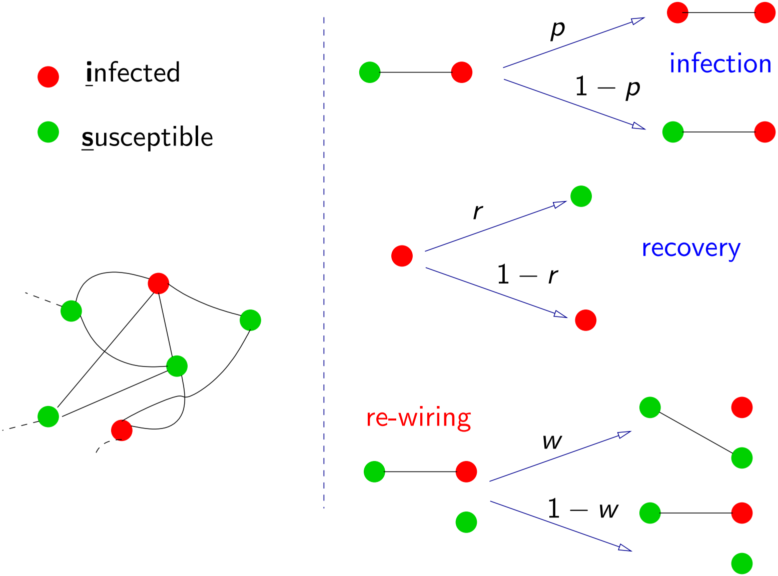
- vertices/nodes $\mathcal{V}$, edges/links $\mathcal{E}$, graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$;
- (weighted) adjacency matrix $A = (a_{ij}(t)) \in \mathbb{R}^{N\times N}$, $i, j \in \{1, 2, \ldots, |\mathcal{V}| =: N\}$;
- vertex state $u_i(t)$; e.g. $u_i(t) \in \{\bullet, \bullet\}$.

node dynamics $\leftrightarrow$ link dynamics

t=2



Example 1: The Adaptive SIS Model (Gross et al)



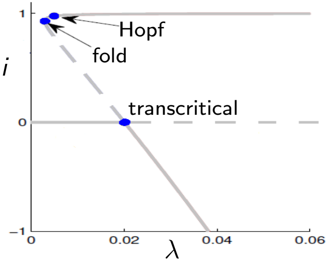
Example 1: The Adaptive SIS Model - New Results*

Use pair-approximation ($l_{abc} = \frac{l_{ab} l_{bc}}{b}$) + noise $\Rightarrow$

$$\begin{aligned} i' &= \lambda\left(\tfrac{\mu}{2} - l_{II} - l_{SS}\right) - ri + \text{noise}, \\ (l_{II})' &= p\left(\tfrac{\mu}{2} - l_{II} - l_{SS}\right)\left(\frac{\frac{\mu}{2} - l_{II} - l_{SS}}{1-i} + 1\right) - 2rl_{II} + \text{noise}, \\ (l_{SS})' &= (r+w)\left(\tfrac{\mu}{2} - l_{II} - l_{SS}\right) - 2\lambda l_{SS}\frac{\left(\frac{\mu}{2} - l_{II} - l_{SS}\right)}{1-i} + \text{noise}. \end{aligned}$$

epidemic $\rightarrow$ healthy

healthy $\rightarrow$ epidemic



upon decreasing λ

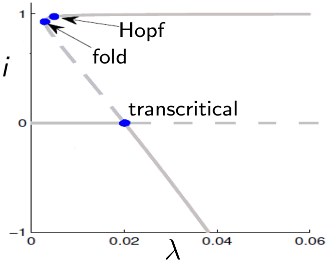
upon increasing λ

* "A mathematical framework for critical transitions: normal forms, variance and applications", **C. Kuehn**, Journal of Nonlinear Science, Vol. 23, No. 3, pp. 457-510, 2013.

Example 1: The Adaptive SIS Model - New Results*

epidemic → healthy — upon decreasing λ

healthy → epidemic — upon increasing λ



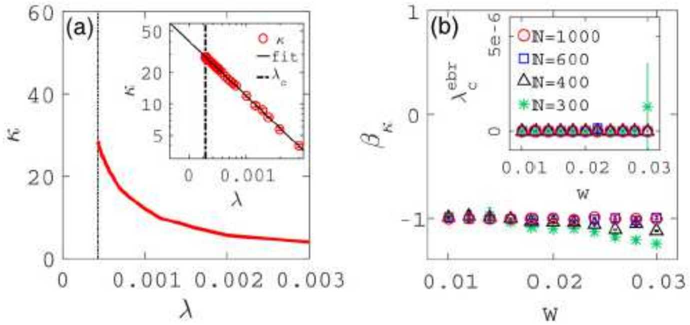
- *early-warning signs* for critical transitions
- *link density* $\text{Var}(l_{SS}(\lambda)) = \mathcal{O}\left((\lambda - \lambda_{\text{tc}})^{-1}\right)$ as $\lambda \to \lambda_{\text{tc}}$.

* "A mathematical framework for critical transitions: normal forms, variance and applications", **C. Kuehn**, Journal of Nonlinear Science, Vol. 23, No. 3, pp. 457-510, 2013.

Example 1: The Adaptive SIS Model - New Results[†]

Idea: utilize network measures as warning signs

Effective branching ratio $\boxed{\kappa := \frac{[SSI]}{[SI]}}$



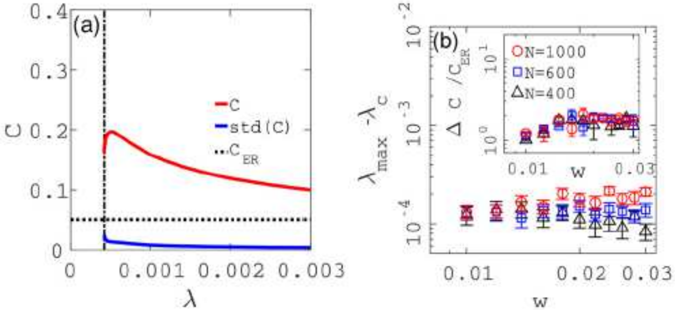
⇒ not a warning sign, despite scaling!

[†] *"Network topology near criticality in adaptive epidemics"*, L. Horstmeyer, **C. Kuehn** and S. Thurner, Physical Review E, Vol 98, 042313, 2018.

Example 1: The Adaptive SIS Model - New Results[†]

Idea: utilize network measures as warning signs

Clustering coefficient $\boxed{C := \frac{\mathrm{Tr}(A^3)}{\sum_{ij}(A^2)_{ij} - \mathrm{Tr}(A^2)}}$



⇒ excellent warning sign, local max!

[†] *"Network topology near criticality in adaptive epidemics"*, L. Horstmeyer, **C. Kuehn** and S. Thurner, Physical Review E, Vol 98, 042313, 2018.

Example 1: The Adaptive SIS Model - New Results†

Idea: utilize network measures as warning signs

Summary (epidemic $\rightarrow$ healthy transition):

- Effective branching ratio $\boxed{\kappa := \frac{[SSI]}{[SI]}}$ ✗
- Clustering coefficient $\boxed{C := \frac{\mathrm{Tr}(A^3)}{\sum_{ij}(A^2)_{ij} - \mathrm{Tr}(A^2)}}$ ✓
- SI link density, triplet densities ✓
- harmonic mean distance ✗
- assortativity ✓
- variance of degree distribution ✗
- eigenvalue gap ✓

† *"Network topology near criticality in adaptive epidemics"*, L. Horstmeyer, **C. Kuehn** and S. Thurner, Physical Review E, Vol 98, 042313, 2018.

Example 2: "Synaptic" Plasticity (Bornholdt-Rohlf)

1. nodes $v_i(t) \in \{\pm 1\}$, directed edges $a_{ij}(t) \in \{-1, 0, +1\}$.
2. dynamical update rule ($t = 0$, random graph), define

$$f_i(t) = \sum_j a_{ij}(t) v_j(t) + \mu v_i(t) + \sigma r_i, \;\; r_i \sim \mathcal{N}(0, 1)$$

$$v_i(t+1) = \begin{cases} \text{sgn}[f_i(t)] & \text{if } f_i(t) \neq 0, \\ v_i(t) & \text{if } f_i(t) = 0. \end{cases}$$

Example 2: "Synaptic" Plasticity (Bornholdt-Rohlf)

1. nodes $v_i(t) \in \{\pm 1\}$, directed edges $a_{ij}(t) \in \{-1, 0, +1\}$.
2. dynamical update rule ($t = 0$, random graph), define

$$f_i(t) = \sum_j a_{ij}(t) v_j(t) + \mu v_i(t) + \sigma r_i, \;\; r_i \sim \mathcal{N}(0,1)$$

$$v_i(t+1) = \begin{cases} \mathrm{sgn}[f_i(t)] & \text{if } f_i(t) \neq 0, \\ v_i(t) & \text{if } f_i(t) = 0. \end{cases}$$

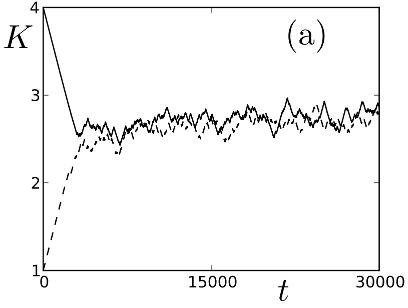
3. T_v node dynamics steps, $T_a := \lfloor T_v/2 \rfloor$, measure `activity`

$$\mathfrak{A}_i := \frac{1}{T_v - T_a} \left[\sum_{t=T_a}^{T_v - 1} v_i(t) \right].$$

4. topological update rule, choose one node i randomly

$$\begin{array}{ll} |\mathfrak{A}_i| > 1 - \delta & \text{create an edge } a_{ij}(t) \neq 0, \\ |\mathfrak{A}_i| \leq 1 - \delta & \text{delete an edge } a_{ij}(t) = 0. \end{array}$$

Example 2: "Synaptic" Plasticity - New Results[‡]



- SOC $=$ steady state near (fast subsystem) bifurcation point
- Optimal finite noise $\leftrightarrow$ steady stochastic resonance (SSR)
- Optimal time scale $\leftrightarrow$ time scale resonance (TSR)

[‡] "Time-scale and noise optimality in self-organized critical adaptive networks", **C. Kuehn**, Physical Review E, Vol. 85, No. 2, 026103, 2012.

Example 3: Adaptive Snowdrift Game (Pacheco et al)

payoff matrix $$M = \begin{pmatrix} b - c/2 & b - c \\ b & 0 \end{pmatrix}$$

- c = cost of cooperation, and b = benefit;
- strategies: cooperation $u_i = C$ or defection $u_i = D$;
- total payoff for node i is $\pi_i = \sum_{j:a_{ij}=1} M_{ij}$;

Example 3: Adaptive Snowdrift Game (Pacheco et al)

$$\text{payoff matrix} \qquad M = \begin{pmatrix} b - c/2 & b - c \\ b & 0 \end{pmatrix}$$

- $c =$ cost of cooperation, and $b =$ benefit;
- strategies: cooperation $u_i = C$ or defection $u_i = D$;
- total payoff for node i is $\pi_i = \sum_{j:a_{ij}=1} M_{ij}$;

Dynamics & Adaptivity:

- $n_u =$ fraction of agents using strategy u, define average payoff

$$\psi(u) := \frac{1}{n_u N} \sum_{i:u_i=u} \pi_i .$$

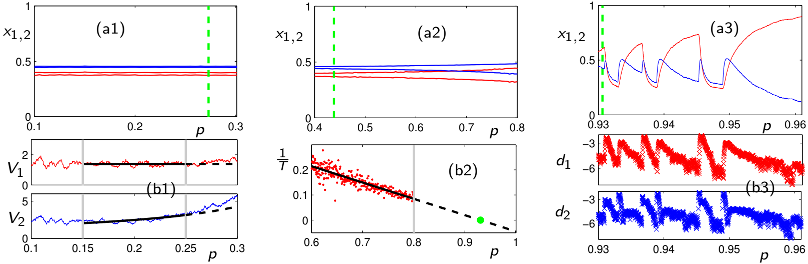
- probability to change dynamics (along a_{ij} edge)

$$f_\beta(i,j) = \left(1 + e^{-\beta[\psi(u_i) - \psi(u_j)]]}\right)^{-1}$$

- re-wire an edge with probability $f_\alpha(i,j)$.

Example 3: Adaptive Snowdrift Game - New Results[§]

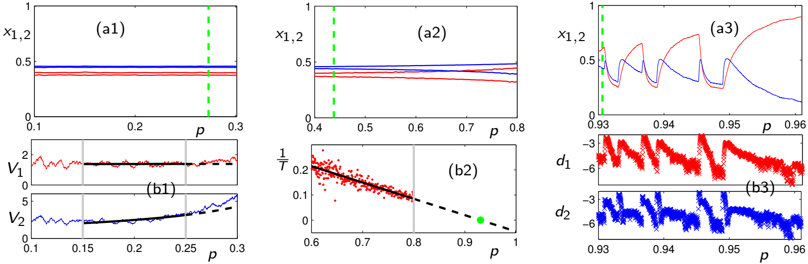
Re-wire (p), adopt ($1-p$) based upon performance.



[§] "Early warning signs for saddle-escape transitions in complex networks", **C. Kuehn**, G. Zschaler and T. Gross, Scientific Reports, Vol. 5, 13190, 2015.

Example 3: Adaptive Snowdrift Game - New Results[§]

Re-wire (p), adopt ($1-p$) based upon performance.



- Hopf bifurcation: variance growth warning
  $\text{Var}(x_{1,2}(p)) = \mathcal{O}\left((p - p_{\text{Hopf}})^{-1}\right)$ as $p \to p_{\text{Hopf}}$.
- Periodic-to-homoclinic transition: period blow-up warning
  $T^{-1}(p) = \mathcal{O}\left((p - p_{\text{hom}})\right)$ as $p \to p_{\text{hom}}$.
- Saddle transition(s): log-distance reduction warning.

[§] "Early warning signs for saddle-escape transitions in complex networks", **C. Kuehn**, G. Zschaler and T. Gross, Scientific Reports, Vol. 5, 13190, 2015.

Example 4: Evolution of Early Life (Jain-Krishna)

1. nodes $v_i(t) \in [0, 1]$, directed edges $a_{ij}(t) \in \{0, 1\}$.
2. dynamical update rule ($t = 0$, random graph), catalytic ODEs

$$v' = A^\top v - \|A^\top v\|_1 v, \quad \|z\|_1 := \sum_{i=1}^{N} |z_i|$$

Example 4: Evolution of Early Life (Jain-Krishna)

1. nodes $v_i(t) \in [0,1]$, directed edges $a_{ij}(t) \in \{0,1\}$.
2. dynamical update rule ($t = 0$, random graph), catalytic ODEs

$$v' = A^\top v - \|A^\top v\|_1 v, \quad \|z\|_1 := \sum_{i=1}^{N} |z_i|$$

3. look at $v(T)$, T "big"; define `weakest` vertex indices

$$\mathcal{J}_* := \{j \in \{1,2,\ldots,N\} : v_j = \min_k v_k\}.$$

4. topological update rule:
   - randomly select $j \in \mathcal{J}_*$,
   - eliminate v_j and all its edges,
   - insert a new vertex, links via Erős-Renyi rule, probability $p \in (0,1)$.

Example 4: Evolution of Early Life - New Results[¶]

Rigorous results - vertex dynamics:

- solutions generically converge to steady state in $\{z \in \mathbb{R}^d : \sum_{i=1}^N z_i = 1\}$
- relation to *projective space* and *algebraic characterization*
- exponential fast convergence $\Rightarrow$ "fast dynamics"

Formal calculations - edge dynamics:

- formation time of the first cycle
- formation time of a single autocatalytic set $(ACS) = \mathcal{G}$
- $Np < 1 \Rightarrow$ "slow dynamics"

$\Rightarrow$ **multiscale / fast-slow system explains spike dynamics**

[¶] "Multiscale dynamics of an adaptive catalytic network", **C. Kuehn**, Mathematical Modelling of Natural Phenomena, Vol. 14, No. 4, 402, 2019.

Example 5: Consensus Networks

- graph Laplacian $L := D - A$ where $D_i = \text{diag}(\sum_j a_{ij})$,
- allow *weights* $\leftrightarrow a_{ij} \in \mathbb{R}$,
- state $v_i(t) \in \mathbb{R}$, consensus

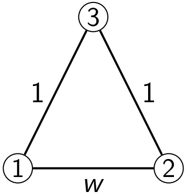
$$v_i(t) = v_j(t) \quad \forall t \geq T.$$

dynamical update rule & topological update rule

$$\begin{aligned} v' &= -Lv = (D(t) - A(t))v, \\ A' &= \varepsilon g(v, A). \end{aligned}$$

Example 5: Consensus Networks - New Results[‖]

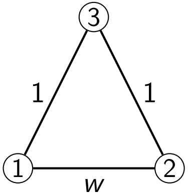
- first results: small-scale "triangle" graph
- methods: fast-slow dynamical systems analysis

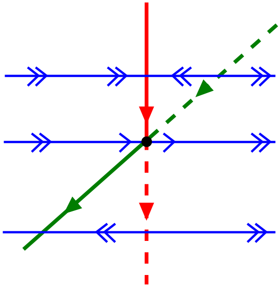


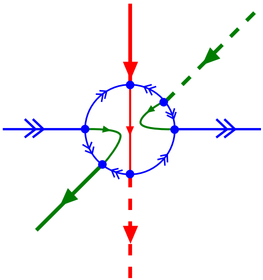
[‖] "On fast-slow consensus networks with a dynamic weight", H. Jardon Kojakhmetov and **C. Kuehn**, arXiv:1904.02690.

Example 5: Consensus Networks - New Results[‖]

- first results: small-scale "triangle" graph
- methods: fast-slow dynamical systems analysis



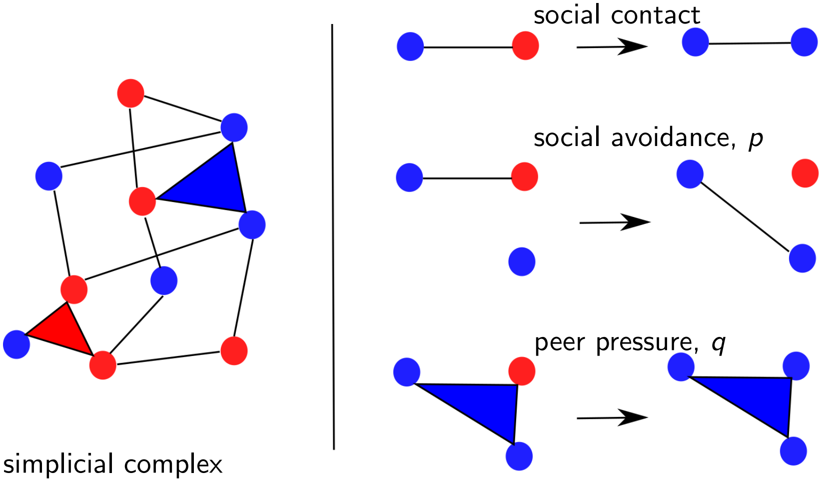




⇒ non-trivial dynamics near consensus manifold!

[‖] "On fast-slow consensus networks with a dynamic weight", H. Jardon Kojakhmetov and **C. Kuehn**, arXiv:1904.02690.

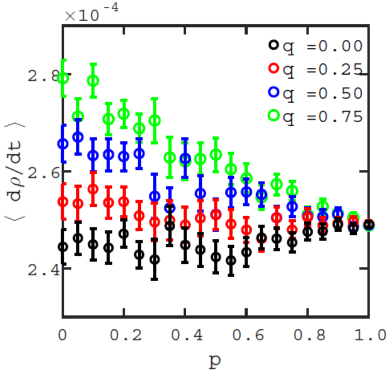
Example 6: Simplicial Adaptive Voters (Horstmeyer-K.)



simplicial complex

Example 6: Simplicial Adaptive Voters - New Results**

- p small ⇒ absorption into one-opinion state
- p large ⇒ fragmentation into two graphs

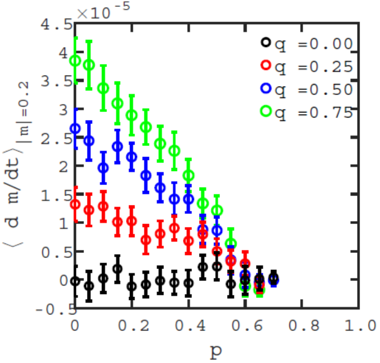


⇒ speed-up via peer-pressure (e.g. edge depletion rate)!

** "An adaptive voter model on simplicial complexes", L. Horstmeyer and **C. Kuehn**, arXiv:1909.05812.

Example 6: Simplicial Adaptive Voters - New Results**

- p small $\Rightarrow$ absorption into one-opinion state
- p large $\Rightarrow$ fragmentation into two graphs



$\Rightarrow$ speed-up via peer-pressure (e.g. drift velocity)!

** "An adaptive voter model on simplicial complexes", L. Horstmeyer and **C. Kuehn**, arXiv:1909.05812.

The Last Slide...

For further information and links to my papers/books:

- http://www.multiscale.systems

The Last Slide...

For further information and links to my papers/books:

- http://www.multiscale.systems

Thank you very much for your attention!!