Optimal Control with Heterogeneous Capitals: Equilibrium Distributions

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In optimal control models with heterogeneous capitals, equilibrium points (or steady states) of the system are functions - “equilibrium distributions”.

Infinite horizon + the dynamics of the system are infinite dimensional + boundary control = a hard problem

Many examples studied in literature share a common feature: there exists an analytic formula for the value function of the C.P.

- age-dependent capitals (Feichtinger et al. 2006, Fabbri-Gozzi 2008),
- spatial growth models (capital stock is heterogeneous in space) (Boucekkine-Camacho-Fabbri 2013, Fabbri 2016, Boucekkine-Fabbri-Federico-Gozzi 2018)
- heterogeneous agents (Moll-Nuno 2018).

Many interesting examples are left out of the picture.
Optimal Investment with Vintage Capital


- Capital accumulation process:

\[
\begin{align*}
\frac{\partial K(\tau, s)}{\partial \tau} &= -\frac{\partial K(\tau, s)}{\partial s} - \mu K(\tau, s) + u_1(\tau, s), \quad (\tau, s) \in ]t, +\infty[ \times ]0, \bar{s}] \\
K(\tau, 0) &= u_0(\tau), \quad \tau \in ]t, +\infty[ \\
K(t, s) &= x(s), \quad s \in [0, \bar{s}]
\end{align*}
\]

(1)

- Profit functional:

\[
I(t, x; u_0, u_1) = \int_t^{+\infty} e^{-\lambda \tau} [R(Q(K(\tau))) - C(u(\tau))] d\tau,
\]

(2)

with

- \( R \) concave, \( C \) convex, both differentiable.
- \( Q(K(\tau)) := \int_0^{\bar{s}} \alpha(s) K(\tau, s) ds = \langle \alpha, K(\tau) \rangle_{L^2} \),
- \( C(u(\tau)) \equiv C_0(u_0(\tau)) + C_1(u_1(\tau)) \equiv C_0(u_0(\tau)) + \int_0^{\bar{s}} c_1(s, u_1(\tau,s)) ds \)

Problem: maximize \( I \) over \((u_0, u_1)\) subject to (1).
General (abstract) Formulation

State space $V'$, with $D(A^*) = V \leftrightarrow H \leftrightarrow V'$. Control space $U$. Both Hilbert.

- **State equation**

  \[
  \begin{cases}
  y'(\tau) = Ay(\tau) + Bu(\tau), & \tau > t \\
  y(t) = x \in V',
  \end{cases}
  \]

  $A$ and $B$ linear, possibly unbounded, $A$ generator of a s,c, semigroup on $V'$.

- **Payoff**

  \[
  J(t, x, u) = \int_{t}^{+\infty} e^{-\lambda\tau} \left[ g_0(y(\tau)) + h_0(u(\tau)) \right] d\tau
  \]

  where $g_0$ and $h_0$ are convex functions.

- **Problem (P)** is minimizing $J_\infty(t, x, u)$ w.r.t. $u$, over the Banach space $L^p_\lambda(t, +\infty; U) = \{ u : [t, +\infty) \to U ; \tau \mapsto u(\tau)e^{-\lambda\tau} \in L^p(t, +\infty; U) \}$.

Optimal investment with vintage capital is of type (P):

\[
V = D(A^*), \quad H = L^2(0, \bar{s}), \quad A = \frac{d}{ds} + \mu I, \quad B(u_0, u_1) = u_1 + \delta_0 u_0
\]

\[
g_0(k) = -R(Q(k)), \quad h_0(u) = C(u)
\]
Range of Application

Common features:

- linear dynamics
- possibly, boundary controls
- general convex/concave payoffs
- possibly, no explicit formula for the value function

Possibly Featured Dynamics (under suitable assumptions on data):

- Vintage capital models, e.g. F.- Gozzi - Kort (submitted)
- Models with time-to-build/delay equations, e.g. F.,-Kort 2020?
- Models with spatial heterogeneity such as Boucekkine- Fabbri-Federico-Gozzi, 2018 and 2019 (A is morally a Laplacian).
General Model - DP

Dynamic Programming (F.-Gozzi, 2010):

- Value function \( Z(t, x) = \inf_{u \in L^p_{\lambda}(t, +\infty; U)} J(t, x, u) = Z(0, x)e^{-\lambda t} \)

- HJB equation (HJB)
  \[
  -\lambda \psi(x) + \langle \psi'(x), Ax \rangle - h^*_0(-B^* \psi'(x)) + g_0(x) = 0,
  \]

- Closed-loop equation (CLE)
  \[
  \begin{cases}
  y'(\tau) = Ay(\tau) + B(h^*_0)'(-B^* \psi'(y(\tau))), & \tau > t \\
  y(t) = x \in V',
  \end{cases}
  \]

Results:

- (Theorem) \( \exists! \) differentiable solution of (HJB) and it is \( \psi(x) = Z(0, x) \).
- (Theorem) Then there exists a unique optimal pair \( (u^*, y^*) \) at \( (t, x) \). The optimal state \( y^* \) is the unique solution of the CLE, and
  \[
  u^*(s) = (h^*_0)'(-B^* \psi'(y^*(s))).
  \]
(Maximum Principle) A co-state $\pi$ is associated to the state, and
(Theorem) NSC optimality conditions are established

$$\begin{align*}
\begin{cases}
y'(\tau) = Ay(\tau) + Bu(\tau), & \tau \geq t \\
y(t) = x
\end{cases},
\pi'(\tau) = (\lambda - A_1^*)\pi(\tau) - g'(y(\tau)), & \tau \geq t \\
\lim_{\tau \to +\infty} e^{(\lambda - \omega)T} \pi(T) = 0, \\
-B^*\pi(\tau) \in \partial h_0(u(\tau)), & \tau \geq t.
\end{align*}$$

(Co-state inclusion) optimal co-state $\pi^*$ coincides with the spatial
gradient of the value function evaluated at optimal state

$$\pi^*(\tau; t, x) = \Psi'(y^*(\tau))$$
Two "types" of equilibrium points:

- stationary solutions of the state-costate system (MP-equilibrium points)

\[
\begin{cases}
x = -A^{-1}Bu \\
\pi = (\lambda - A_1^*)^{-1}g_0'(x), \\
u = (h_0^*)'(-B^*\pi).
\end{cases}
\]

- stationary solutions of the CLE (CL-equilibrium points)

\[
Ax + B(h_0^*)'(-B^*\Psi'(x)) = 0.
\]

(Theorem) From the relationship between CLE/MP (and when \(A\) invertible) -
equilibrium points are fixed points of the operator \(T : V' \rightarrow V'\)

\[
Tx := -A^{-1}B(h_0^*)'(-B^*(\lambda - A_1^*)^{-1}g_0'(x)).
\]

(Theorem- Stability) CL-Equilibrium points are stable/asymptotically stable
(under suitable assumptions) in the topology of \(V'\).
Corollary of the general NSC of optimality:

\((K^*, u^*)\) optimal iff \(\exists\) a costate \(\zeta^*\) such that \((K^*, \zeta^*, u^*)\) satisfy:

\[
\begin{align*}
  u_0^*(\tau) &= (C_0^*)'(\zeta^*(\tau, 0)), \\
  u_1^*(\tau, s) &= ((C_1^*)'(\zeta^*(\tau, \cdot)))[s] = [c_1(s, \cdot)^*]'(\zeta^*(\tau, s))
\end{align*}
\]

\[
\zeta^*(\tau, s) = \int_s^{\bar{s}} e^{-(\lambda+\mu)(\xi-s)} R' \left( \int_0^{\bar{s}} \alpha(\theta) K^*(\tau + \xi - s, \theta) d\theta \right) \alpha(\xi) d\xi
\]

\[
\lim_{T \to +\infty} e^{(\lambda-\omega)T} \zeta(T, s) = 0, \text{ a.a. } s \in [0, \bar{s}].
\]

\[
K^*(\tau, s) = \begin{cases}
  e^{-\mu(\tau-s)} x(s - \tau + t) + \int_0^{\tau-t} e^{-\mu \sigma} u_1^*(\tau - \sigma, s - \sigma) d\sigma & s \in [\tau - t, \bar{s}], \tau \in [t, \bar{s} + t] \\
  e^{-\mu(\tau-s)} u_0^*(\tau - s) + \int_0^{s} e^{-\mu \sigma} u_1^*(\tau - \sigma, s - \sigma) d\sigma & s \in [0, \tau - t], \tau \in [t, \bar{s} + t] \\
  0 & s \in [0, \bar{s}], \tau \in (\bar{s} + t, +\infty)
\end{cases}
\]
Finding equilibrium points is equivalent to solving a numeric equation in $\eta$.

- $\bar{\alpha}(s) = \int_s^\bar{s} e^{-(\mu+\lambda)(\sigma-s)}\alpha(\sigma)d\sigma$

- For $\eta \in \mathbb{R}$
  
  $F(\eta)[s] = (C_0^*)'(\eta\bar{\alpha}(0)) e^{-\mu s} + \int_0^s e^{-\mu(s-\sigma)}[c_1(\sigma, \cdot)^*]'(\eta\bar{\alpha}(\sigma)) d\sigma$,

- $\bar{\eta}$ a solution in $\mathbb{R}$ of
  
  $\eta = R'(\langle \alpha, F(\eta) \rangle)$.

- (Theorem) All equilibrium points are of type
  
  $\bar{k}(s) = F(\bar{\eta})[s]$. 
Specializing Formulas I

With Linear-quadratic costs

\[ C(u) = \int_0^{\tilde{s}} \left[ \beta_1(s)u_1^2(s) + q_1(s)u_1(s) \right] ds + \beta_0 u_0^2 + q_0 u_0 \]

We set

- \( w_1(s) = \frac{\bar{\alpha}(0)}{2\beta_0} e^{-\mu s} + \int_0^s e^{-\mu(s-\sigma)} \frac{\bar{\alpha}(\sigma)}{2\beta_1(\sigma)} d\sigma \)
- \( w_2(s) = \frac{q_0}{2\beta_0} e^{-\mu s} + \int_0^s e^{-\mu(s-\sigma)} \frac{q_1(\sigma)}{2\beta_1(\sigma)} d\sigma \)
- \( c_1 = \langle w_1, \alpha \rangle = \int_0^{\tilde{s}} \alpha(s)w_1(s)ds, \quad c_2 = \langle w_2, \alpha \rangle = \int_0^{\tilde{s}} \alpha(s)w_2(s)ds. \)

Then

- if \( \bar{\eta} \) solves \( \eta = R'(\eta c_1 - c_2) \), then
  \[ \tilde{k}(s) = -w_2(s) + \bar{\eta}w_1(s) \]

(Other choices of treatable costs:

- Linear+quadratic with constrained control
- Linear+Power costs)
Specializing Formulas II

Different Revenues (+Linear-quadratic costs)

- $R(Q) = -aQ^2 + bQ$, then
  \[ \bar{\eta} = \frac{2ac_2 + b}{1 + 2ac_1} \]

- $R(Q) = \ln(1 + Q)$, for $Q \geq 0$ (and $R(Q) = Q$ for $Q < 0$), then
  \[ \bar{\eta} = \frac{\sqrt{(1 - c_2)^2 + 4c_1} - (1 - c_2)}{2c_1} \]

- $R(Q) = b[(\nu + Q)^\gamma - \nu]$, with $\gamma \in (0, 1)$, $b, \nu > 0$, for $Q \geq 0$ and $R(Q) = \gamma \nu^{\gamma-1}Q$ for $Q < 0$, then $\bar{\eta}$ is the unique positive solution of
  \[ \eta = \frac{b\gamma}{(\nu + c_1\eta - c_2)^{1-\gamma}} \]

- $R(Q) = bQ^\gamma$, with $\gamma \in (0, 1)$, $b > 0$, for $Q \geq 0$, and $R(Q) = -\infty$ for $Q < 0$ (case with state constraints) then $\bar{\eta}$ is the unique positive solution of
  \[ \eta = \frac{b\gamma}{(c_1\eta - c_2)^{1-\gamma}}. \]
Different choices of coefficient

- \( \alpha(s) \equiv \alpha, \quad \beta_1(s) \equiv \beta_0, \quad q_1(s) = q_0 e^{-ws} \)
- \( R(Q) = b ((\theta + Q)\gamma - \theta) \)

By specializing the coefficient one

- obtains more explicit formulas for optimal stationary couples
- can perform sensitivity analysis with respect to different parameters, (e.g. show that in both cases \( \bar{k} \) is increasing w.r.t. the productivity parameter \( \alpha \) iff \( \alpha \leq \alpha^* \), and that optimal control and trajectories are hump-shaped).
Thank You!