# Optimal Control with Heterogeneous Capitals: Equilibrium Distributions

S. Faggian,[1] F. Gozzi,[2] P. Kort[3]

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# Introduction

- In optimal control models with heterogeneous capitals, equilibrium points (or steady states) of the system are functions - "equilibrium distributions".
- infinite horizon + the dynamics of the system are infinite dimensional + boundary control = a hard problem
- many examples studied in literature share a common feature: *there exists an analytic formula for the value function of the C.P.*
  - age-dependent capitals (Feichtinger et al. 2006, Fabbri-Gozzi 2008),
  - spatial growth models (capital stock is heterogeneous in space) (Boucekkine-Camacho-Fabbri 2013, Fabbri 2016, Boucekkine-Fabbri-Federico-Gozzi 2018)
  - growth models with time-to-build (Asea-Zak 1999, Bambi 2008, Bambi-Fabbri-Gozzi 2012),
  - heterogeneous agents (Moll-Nuno 2018).
- Many interesting examples are left out of the picture.

# Optimal Investment with Vintage Capital

*[Barucci-Gozzi 1998-2001; Feichtinger et al. 2003-2006, F. 2004-2008, F. and Gozzi 2005-2010]*

- Capital accumulation process:

$$
\begin{cases}
\frac{\partial K(\tau,s)}{\partial \tau} = -\frac{\partial K(\tau,s)}{\partial s} - \mu K(\tau,s) + u_1(\tau,s), & (\tau,s) \in ]t,+\infty[\times]0,\bar{s}] \\
K(\tau,0) = u_0(\tau), & \tau \in ]t,+\infty[ \\
K(t,s) = x(s), & s \in [0,\bar{s}]
\end{cases}
\tag{1}
$$

- Profit functional:

$$
I(t,x;u_0,u_1) = \int_t^{+\infty} e^{-\lambda\tau}[R(Q(K(\tau)) - C(u(\tau))]d\tau, \tag{2}
$$

with

- $R$ concave, $C$ convex, both differentiable.
- $Q(K(\tau)) := \int_0^{\bar{s}} \alpha(s)K(\tau,s)ds = \langle \alpha, K(\tau) \rangle_{L^2}$,
- $C(u(\tau)) \equiv C_0(u_0(\tau)) + C_1(u_1(\tau)) \equiv C_0(u_0(\tau)) + \int_0^{\bar{s}} c_1(s,u_1(\tau,s))ds$

Problem: maximize $I$ over $(u_0, u_1)$ subject to (1).

# General (abstract) Formulation

State space $V'$, with $D(A^*) = V \hookrightarrow H \hookrightarrow V'$. Control space $U$. Both Hilbert.

- State equation
$$\begin{cases} y'(\tau) = Ay(\tau) + Bu(\tau), & \tau > t \\ y(t) = x \in V', \end{cases}$$
$A$ and $B$ linear, possibly unbounded, $A$ generator of a s,c, semigroup on $V'$.
- Payoff
$$J(t,x,u) = \int_t^{+\infty} e^{-\lambda\tau} \left[g_0\left(y(\tau)\right) + h_0\left(u(\tau)\right)\right] d\tau$$
where $g_0$ and $h_0$ are convex functions.
- Problem (P) is minimizing $J_\infty(t,x,u)$ w.r.t. $u$, over the Banach space $L^p_\lambda(t,+\infty;U) = \{u : [t,+\infty) \to U\ ;\ \tau \mapsto u(\tau)e^{-\frac{\lambda\tau}{p}} \in L^p(t,+\infty;U)\}$.

Optimal investment with vintage capital is of type (P):

$$V = D(A^*),\quad H = L^2(0,\bar{s}),\quad A = \frac{d}{ds} + \mu I,\quad B(u_0,u_1) = u_1 + \delta_0 u_0$$
$$g_0(k) = -R\left(Q(k)\right),\quad h_0(u) = C(u)$$

# Range of Application

Common features:

- linear dynamics
- possibly, boundary controls
- general convex/concave payoffs
- possibly, no explicit formula for the value function

Possibly Featured Dynamics (under suitable assumptions on data):

- Vintage capital models, e.g. F.- Gozzi - Kort (submitted)
- Models with time-to-build/delay equations, e.g. F.,-Kort 2020?


- Models with spatial heterogeneity such as Boucekkine- Fabbri-Federico-Gozzi, 2018 and 2019 ($A$ is morally a Laplacian).

# General Model - DP

Dynamic Programming (F.-Gozzi, 2010):

- Value function $Z(t,x) = \inf_{u \in L^p_\lambda(t,+\infty;U)} J(t,x,u) = Z(0,x)e^{-\lambda t}$
- HJB equation (HJB)

$$-\lambda\psi(x) + \langle \psi'(x), Ax \rangle - h_0^*(-B^*\psi'(x)) + g_0(x) = 0,$$

- Closed-loop equation (CLE)

$$\begin{cases} y'(\tau) = Ay(\tau) + B(h_0^*)'(-B^*\psi'(y(\tau))), & \tau > t \\ y(t) = x \in V', \end{cases}$$

Results:

- (Theorem) $\exists!$ differentiable solution of (HJB) and it is $\Psi(x) = Z(0,x)$.
- (Theorem) Then there exists a unique optimal pair $(u^*, y^*)$ at $(t,x)$. The optimal state $y^*$ is the unique solution of the CLE, and

$$u^*(s) = (h_0^*)'(-B^*\Psi'(y^*(s))).$$

# General Model - MP

F.-Gozzi-Kort (submitted 2019):

- (Maximum Principle) A co-state $\pi$ is associated to the state, and (Theorem) NSC optimality conditions are established

$$\begin{cases} y'(\tau) = Ay(\tau) + Bu(\tau), & \tau \geq t \\ y(t) = x \\ \pi'(\tau) = (\lambda - A_1^*)\pi(\tau) - g_0'(y(\tau)), & \tau \geq t \\ \lim_{T \to +\infty} e^{(\lambda-\omega)T}\pi(T) = 0, \\ -B^*\pi(\tau) \in \partial h_0(u(\tau)), & \tau \geq t. \end{cases}$$

- (Co-state inclusion) optimal co-state $\pi^*$ coincides with the spatial gradient of the value function evaluated at optimal state

$$\pi^*(\tau; t, x) = \Psi'(y^*(\tau))$$

# General Model - Equilibrium distributions

Two "types" of equilibrium points:

- stationary solutions of the state-costate system (MP-equilibrium points)

$$\begin{cases} x = -A^{-1}Bu \\ \pi = (\lambda - A_1^*)^{-1} g_0'(x), \\ u = (h_0^*)'(-B^*\pi)). \end{cases}$$

- stationary solutions of the CLE (CL-equilibrium points)

$$Ax + B(h_0^*)'(-B^*\Psi'(x)) = 0.$$

- (Theorem) From the relationship between CLE/MP (and when $A$ invertible) - equilibrium points are fixed points of the operator $T : V' \to V'$

$$Tx := -A^{-1}B(h_0^*)'(-B^*(\lambda - A_1^*)^{-1} g_0'(x)).$$

- (Theorem- Stability) CL-Equilibrium points are stable/asymptotically stable (under suitable assumptions) in the topology of $V'$.

Corollary of the general NSC of optimality:

$(K^*, u^*)$ optimal iff $\exists$ a costate $\zeta^*$ such that $(K^*, \zeta^*, u^*)$ satisfy:

$$u_0^*(\tau) = (C_0^*)'(\zeta^*(\tau, 0)), \quad u_1^*(\tau, s) = \big((C_1^*)'(\zeta^*(\tau, \cdot))\big)[s] = [c_1(s, \cdot)^*]'(\zeta^*(\tau, s))$$

$$\zeta^*(\tau, s) = \int_s^{\bar{s}} e^{-(\lambda+\mu)(\xi - s)} R' \left( \int_0^{\bar{s}} \alpha(\theta) K^*(\tau + \xi - s, \theta) d\theta \right) \alpha(\xi) d\xi$$

$$\lim_{T \to +\infty} e^{(\lambda - \omega)T} \zeta(T, s) = 0, \ a.a.\ s \in [0, \bar{s}].$$

$$K^*(\tau, s) = \begin{cases} e^{-\mu(\tau - s)} x(s - \tau + t) + \int_0^{\tau - t} e^{-\mu\sigma} u_1^*(\tau - \sigma, s - \sigma) d\sigma & s \in [\tau - t, \bar{s}],\ \tau \in [t, \bar{s} + t] \\ e^{-\mu(\tau - s)} u_0^*(\tau - s) + \int_0^{s} e^{-\mu\sigma} u_1^*(\tau - \sigma, s - \sigma) d\sigma & s \in [0, \tau - t],\ \tau \in [t, \bar{s} + t] \\ 0 & s \in [0, \bar{s}],\ \ \tau \in (\bar{s} + t, +\infty) \end{cases}$$

# Construction of Equilibrium Distributions

Finding equilibrium points is equivalent to solving a numeric equation in $\eta$.

- $\bar{\alpha}(s) = \int_s^{\bar{s}} e^{-(\mu+\lambda)(\sigma-s)}\alpha(\sigma)d\sigma$
- for $\eta \in \mathbb{R}$
  $F(\eta)[s] = (C_0^*)'(\eta\bar{\alpha}(0))\, e^{-\mu s} + \int_0^s e^{-\mu(s-\sigma)}[c_1(\sigma,\cdot)^*]'(\eta\bar{\alpha}(\sigma))\, d\sigma,$
- $\bar{\eta}$ a solution in $\mathbb{R}$ of
  $$\eta = R'(\langle \alpha, F(\eta)\rangle).$$
- (Theorem) All equilibrium points are of type
  $$\bar{k}(s) = F(\bar{\eta})[s].$$

# Specializing Formulas I

## With Linear-quadratic costs

$$C(u) = \int_0^{\bar{s}} [\beta_1(s)u_1^2(s) + q_1(s)u_1(s)]ds + \beta_0 u_0^2 + q_0 u_0$$

We set

- $w_1(s) = \frac{\bar{\alpha}(0)}{2\beta_0} e^{-\mu s} + \int_0^s e^{-\mu(s-\sigma)} \frac{\bar{\alpha}(\sigma)}{2\beta_1(\sigma)} d\sigma$
- $w_2(s) = \frac{q_0}{2\beta_0} e^{-\mu s} + \int_0^s e^{-\mu(s-\sigma)} \frac{q_1(\sigma)}{2\beta_1(\sigma)} d\sigma$
- $c_1 = \langle w_1, \alpha \rangle = \int_0^{\bar{s}} \alpha(s) w_1(s) ds, \quad c_2 = \langle w_2, \alpha \rangle = \int_0^{\bar{s}} \alpha(s) w_2(s) ds.$

Then

- if $\bar{\eta}$ solves $\eta = R'(\eta c_1 - c_2)$, then

$$\bar{k}(s) = -w_2(s) + \bar{\eta} w_1(s)$$

(Other choices of treatable costs:

- Linear+quadratic with constrained control
- Linear+Power costs )

# Specializing Formulas II

## Different Revenues (+Linear-quadratic costs)

- $R(Q) = -aQ^2 + bQ$, then
$$\bar{\eta} = \frac{2ac_2 + b}{1 + 2ac_1}$$
- $R(Q) = \ln(1 + Q)$, for $Q \geq 0$ (and $R(Q) = Q$ for $Q < 0$), then
$$\bar{\eta} = \frac{\sqrt{(1 - c_2)^2 + 4c_1} - (1 - c_2)}{2c_1}$$
- $R(Q) = b[(\nu + Q)^\gamma - \nu]$, with $\gamma \in (0, 1)$, $b, \nu > 0$, for $Q \geq 0$ and $R(Q) = \gamma \nu^{\gamma - 1} Q$ for $Q < 0$, then $\bar{\eta}$ is the unique positive solution of
$$\eta = \frac{b\gamma}{(\nu + c_1\eta - c_2)^{1-\gamma}}$$
- $R(Q) = bQ^\gamma$, with $\gamma \in (0, 1)$, $b > 0$, for $Q \geq 0$, and $R(Q) = -\infty$ for $Q < 0$ (case with state constraints) then $\bar{\eta}$ is the unique positive solution of
$$\eta = \frac{b\gamma}{(c_1\eta - c_2)^{1-\gamma}}.$$

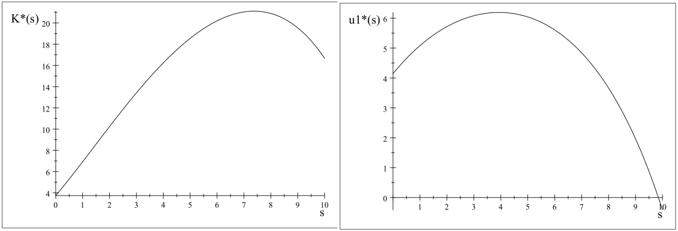
# Specializing Formulas III

## Different choices of coefficient

- $\alpha(s) \equiv \alpha, \quad \beta_1(s) \equiv \beta_0, \quad q_1(s) = q_0 e^{-ws}$
- $R(Q) = b\left((\theta + Q)^\gamma - \theta\right)$

By specializing the coefficient one

- obtains more explicit formulas for optimal stationary couples
- can perform sensitivity analysis with respect to different parameters, (e.g.show that in both cases $\bar{k}$ is increasing w.r.t. the productivity parameter $\alpha$ iff $\alpha \leq \alpha^*$, and that optimal control and trajectories are hump-shaped).



Thank You!