

Similarities and Differences Between Two-Particle and Three-Particle Interference

DANIEL M. GREENBERGER,

City College of the City University of New York
New York, New York 10031

MICHAEL HORNE

Stonehill College, North Easton, Massachusetts 02357

ANTON ZEILINGER

Institut für Experimentalphysik
Universität Innsbruck, A-6020 Innsbruck, Austria

Abstract

We illustrate various methods for implementing experiments that split particles into three beams, using “tritters”, or use three coherent particles (GHZ states), in order to illustrate our belief that any experiment that can be done using two particles is more interesting with three particles.

Introduction

There have been various schemes recently proposed to create three-particle superpositions as entangled or GHZ states [1]. One obvious use of them would be to implement the GHZ theorem [2], which would go beyond two-particle Bell theorems. But it would seem that there should be other uses for such superpositions. Our basic position is that there are things that can be done with three-particle states that cannot be done with two-particle states, and that anything you can do with two-particle states can be done more interestingly with three-particle states. In this report, we shall illustrate with a few examples.

Beam Splitters and Tritters

A beam splitter has two input beams and two output beams, connected by a unitary transformation. The simplest case is shown in Fig. 1. Here we have

$$\begin{aligned} |a\rangle &\rightarrow \frac{1}{\sqrt{2}} (|d\rangle + i|c\rangle), \\ |b\rangle &\rightarrow \frac{1}{\sqrt{2}} (|c\rangle + i|d\rangle). \end{aligned} \tag{1}$$

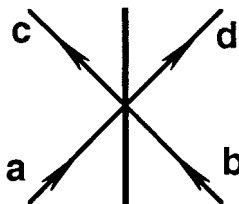


Fig. 1: A simple Beam Splitter. The input ports are a and b , and the output ports are c and d .

We can also write this as

$$\begin{pmatrix} c \\ d \end{pmatrix} = R \begin{pmatrix} a \\ b \end{pmatrix}, \quad R = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad RR^\dagger = 1. \tag{2}$$

The three beam extension of this is called a “tritter” (the general n -beam case is called a “critter”). One can construct any finite unitary matrix from a combination of critters, mirrors, and phase shifters [3]. The critter itself is constructed of beam splitters, mirrors, and phase shifters.

The tritter is shown in Fig. 2a. A simple case where the ports have equal amplitudes can be constructed from the cube roots of unity, as shown in Fig. 2b. These roots, $1, \lambda, \mu$, have the following properties:

$$\begin{aligned} \lambda &= e^{2\pi i/3}, & \mu &= e^{4\pi i/3}, \\ 1^3 &= \lambda^3 = \mu^3 = 1, \\ \lambda^2 &= \mu, & \mu^2 &= \lambda, & \lambda\mu &= 1, \\ 1 + \lambda + \mu &= 0. \end{aligned} \tag{3}$$

The tritter has the properties:

$$\begin{aligned} |a\rangle &\rightarrow \frac{1}{\sqrt{3}} (|d\rangle + |e\rangle + |f\rangle), \\ |b\rangle &\rightarrow \frac{1}{\sqrt{3}} (|d\rangle + \lambda |e\rangle + \mu |f\rangle), \\ |c\rangle &\rightarrow \frac{1}{\sqrt{3}} (|d\rangle + \mu |e\rangle + \lambda |f\rangle), \end{aligned} \tag{4}$$

$$R = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \lambda & \mu \\ 1 & \mu & \lambda \end{pmatrix}, \quad RR^\dagger = 1.$$

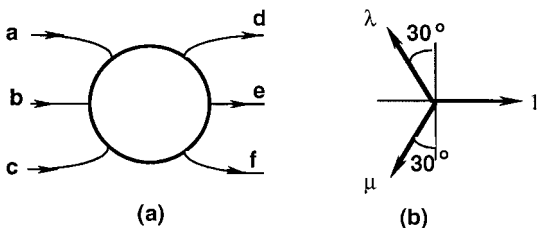


Fig. 2: The Tritter. (a) A simple tritter. The input ports are a, b , and c , and the output ports are d, e , and f . They are related by a unitary transformation. (b) the cube roots of unity in the complex plane, labeled, $1, \lambda$, and μ .

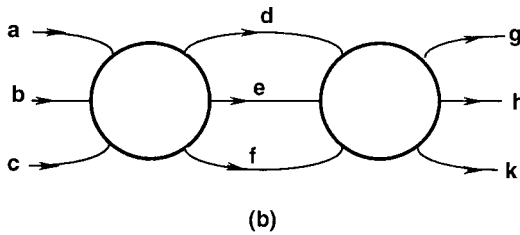
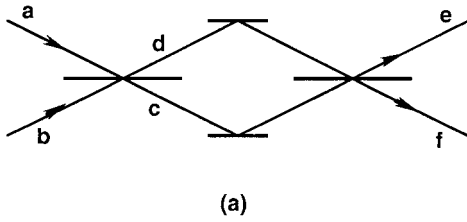


Fig. 3: Multiport Interferometers. (a) Two beam splitters are needed for two input ports. Beam a emerges at f , and beam b at e . (b) Two tritters are needed for three input ports. Beam a emerges at g , beam b at k , and beam c at h (note the inversion).

If one has an interferometer of two beam splitters, as in Fig. 3a, then

$$\begin{aligned} |b\rangle &\rightarrow i |e\rangle, \\ |a\rangle &\rightarrow i |f\rangle, \end{aligned} \tag{5}$$

so that it is in a sense “diagonal”. The same is true for our tritter. If one has an interferometer like that shown in Fig. 3b, then

$$|a\rangle \rightarrow |g\rangle, \quad |b\rangle \rightarrow |k\rangle, \quad |c\rangle \rightarrow |h\rangle. \tag{6}$$

Notice here that there is a slight interchange, in that $|b\rangle \rightarrow |k\rangle$, rather than $|b\rangle \rightarrow |h\rangle$.

Two Particle and Three-Particle Systems

One can use beam-splitters and tritters to construct two-particle and three-particle interference systems. The two-particle case is shown in Fig. 4, where

$$\begin{aligned} \psi &= \frac{1}{\sqrt{2}} (|a\rangle |a'\rangle + |b\rangle |b'\rangle) \rightarrow \frac{1}{\sqrt{2}} (|a\rangle |a'\rangle e^{i\alpha} + |b\rangle |b'\rangle) \\ &\rightarrow \frac{i e^{i\alpha/2}}{\sqrt{2}} \left((|c\rangle |d'\rangle + |d\rangle |c'\rangle) \cos \frac{\alpha}{2} + (|c\rangle |c'\rangle - |d\rangle |d'\rangle) \sin \frac{\alpha}{2} \right). \end{aligned} \tag{7}$$

The three-particle case, assuming that the source S is in a GHZ state, is shown in Fig. 5. In this case, one has

$$\psi = \frac{1}{\sqrt{3}} (|a\rangle |a'\rangle |a''\rangle + |b\rangle |b'\rangle |b''\rangle + |c\rangle |c'\rangle |c''\rangle). \tag{8}$$

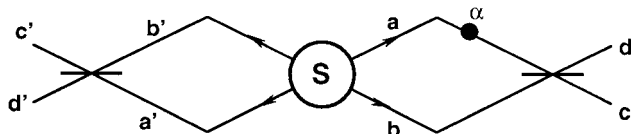


Fig. 4: A two-entangled-particle Interferometer. A phase shifter α is placed in one of the beams. Interference is seen by counting coincidences between the two particles (primed and unprimed). No individual counter will see any interference effects.

Each of the photons passes through its respective tritter, so that

$$|a\rangle \rightarrow \frac{1}{\sqrt{3}} (|d\rangle + |e\rangle + |f\rangle),$$

$$|a'\rangle \rightarrow \frac{1}{\sqrt{3}} (|d'\rangle + |e'\rangle + |f'\rangle),$$

etc.

There are three phase shifters present, in the beams a , b , and c . At the output, one counts coincidences between the three photons. There are 27 possible sets of coincidences between them, one unprimed, one single-primed, and one double-primed, e.g. $dd'd''$, $dd'e''$, $ef'd''$, etc. However, just as the two-particle case simplified, so that the 4 possible output probabilities each reduced to one of the functions $\cos^2 \alpha/2$, $\sin^2 \alpha/2$, so too in the three particle case the output probability of each coincidence set reduces to one of three possible functions.

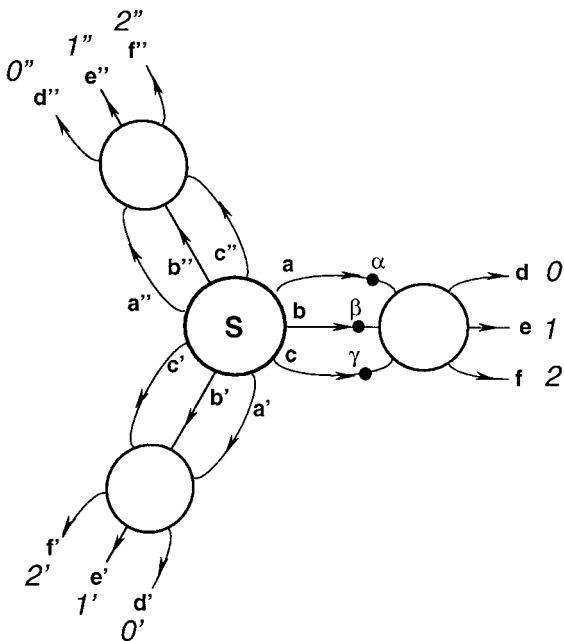


Fig. 5: A Three-entangled-particle Interferometer. Three phase shifters α , β , and γ , are placed in the system. Interference is seen by counting coincidences between all three particles. Individual particles and pairs of particles will show no interference.

These functions are

$$\begin{aligned}
 f_0(\alpha, \beta, \gamma) &= \frac{1}{27} |e^{i\alpha} + e^{i\beta} + e^{i\gamma}|^2, \\
 f_1(\alpha, \beta, \gamma) &= \frac{1}{27} |e^{i\alpha} + \lambda e^{i\beta} + \mu e^{i\gamma}|^2 = f_0(\alpha, \beta + 2\pi/3, \gamma - 2\pi/3), \\
 f_2(\alpha, \beta, \gamma) &= \frac{1}{27} |e^{i\alpha} + \mu e^{i\beta} + \lambda e^{i\gamma}|^2 = f_0(\alpha, \beta - 2\pi/3, \gamma + 2\pi/3), \\
 |f_0|^2 + |f_1|^2 + |f_2|^2 &= \frac{1}{9}.
 \end{aligned}
 \tag{10}$$

How can one tell which coincidence output goes with which function? In Fig. 5, we have given a number to each output,

$$d = d' = d'' = 0, \quad e = e' = e'' = 1, \quad f = f' = f'' = 2.
 \tag{11}$$

Then, to find the probability for the output of any coincidence set, one just adds the three numbers, mod 3, and the sum gives the correct function f_i . For example,

$$\begin{aligned}
 P_{dd'd''} &= f_{0+0+0} = f_0, \\
 P_{de'f''} &= f_{0+1+2} = f_{0 \pmod{3}} = f_0, \\
 P_{ef'f''} &= f_{1+2+2} = f_{2 \pmod{3}} = f_2, \\
 &\text{etc.}
 \end{aligned}
 \tag{12}$$

The function f_0 is given by

$$\begin{aligned}
 f_0 &= \frac{1}{27} |e^{i\alpha} + e^{i\beta} + e^{i\gamma}|^2 \\
 &= \frac{1}{27} \left(3 + 2 \cos \frac{\alpha - \beta}{2} + 2 \cos \frac{\beta - \gamma}{2} + 2 \cos \frac{\gamma - \alpha}{2} \right) \\
 &= \frac{1}{27} \left(1 + 8 \cos \frac{\alpha - \beta}{2} \cos \frac{\beta - \gamma}{2} \cos \frac{\gamma - \alpha}{2} \right).
 \end{aligned}
 \tag{13}$$

It is a function of two variables, $x = \alpha - \beta$, $y = \beta - \gamma$, and one can write it

$$\begin{aligned}
 f_0(x, y) &= \frac{1}{27} \left(1 + 8 \cos \frac{x}{2} \cos \frac{y}{2} \cos \frac{x+y}{2} \right), \\
 f_0(0, 0) &= f_0(0, 2\pi) = f_0(2\pi, 0) = f_0(2\pi, 2\pi) = \frac{1}{3}, \\
 f_0(2\pi/3, 2\pi/3) &= f_0(4\pi/3, 4\pi/3) = 0, \\
 f_0(x, \pi) &= f_0(\pi, y) = \frac{1}{27}.
 \end{aligned}
 \tag{14}$$

It is periodic in both x and y with a period of 2π .

In three-particle interference experiments, one can see from eq. (13), that if any two of the angles, say β and γ differ by π , then the function f_0 becomes constant, independent of the other angle α . Thus one of the beams can be used to monitor the other two, and to determine whether they can interfere or not. This is not possible with two-particle interference.

How a Tritter Can Be Better Than a Beam-Splitter

We will point out a sample situation where a tritter can replace a beam-splitter and perform a better job. But since there are three beams within a tritter, there are obviously many circumstances where one can choose either one path, which will destroy coherence, or the other two as a group, which will then not fully destroy coherence. There are many possibilities, of which ours is merely the simplest.

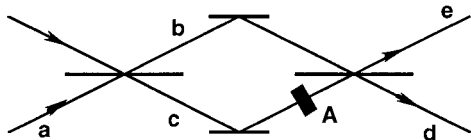
We will examine the case of the Vaidman-Elitzur bomb [4], merely replacing the “bomb” by an absorber, so that if the photon strikes the absorber, it will have a 100% chance of being absorbed (no explosion!). The original situation is shown in Fig. 6a. There *may* be an absorber in the beam c at A , and the problem is to determine by passing one photon through the system, without having the beam absorbed by the absorber, whether the absorber is there or not.

Classically the problem is insoluble. Quantum mechanically, if the absorber is missing, the wave function of a photon sent into the system at a will evolve as

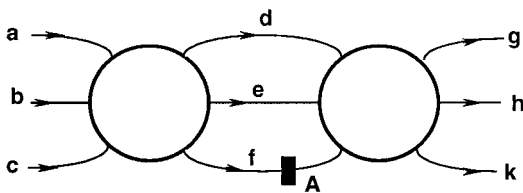
$$|a\rangle \rightarrow \frac{1}{\sqrt{2}} (|b\rangle + i|c\rangle) \rightarrow \frac{1}{2} ((|d\rangle + i|e\rangle) + i(|e\rangle + i|d\rangle)) = i|e\rangle. \tag{15}$$

Thus, the photon will never show up in the detector at d . But if the absorber is present, the photon will evolve as

$$|a\rangle \rightarrow \frac{1}{\sqrt{2}} (|b\rangle + i|c\rangle) \rightarrow \frac{1}{\sqrt{2}} \left((|b\rangle + i|A\rangle) \rightarrow \frac{1}{2} (|d\rangle + i|e\rangle) \right) + \frac{1}{\sqrt{2}} |A\rangle. \tag{16}$$



(a)



(b)

Fig. 6: Detecting a Particle Without Interacting With It. (a) The two-particle version has an absorber at A , whose presence can be detectable by counting a photon at d that entered the system at a , and that has never interacted with it. (b) The three-particle version. The absorber at A can be detected by counting a photon at h that entered the system in a specified way through ports a and b .

Here, the state $|A\rangle$ denotes that the photon has been absorbed at A . Therefore, there is a 50% chance that the photon will be absorbed, a 25% chance that it will show up at the detector at e , and now a 25% chance that it will show up at the detector at d . Since if the absorber were absent, it would never show up at d , this means that whenever the counter d clicks, the absorber *must* have been present in the system. If it is present, half the time the photon will not get through. But if it does get through, then half the time it will be detectable by the clicking of counter d . In these cases, we will have detected the presence of an absorber in the system, without having ever interacted with the absorber! This is an amazing demonstration of the non-locality of quantum mechanics.

However, the statistics can be improved using instead a tritter. In Fig. 6b, if the absorber is absent from beam f , then from eq. (6) one sees that an initial wave function $|\psi\rangle = \frac{1}{\sqrt{2}} (|a\rangle + \lambda |b\rangle)$ will evolve into

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|a\rangle + \lambda |b\rangle) \rightarrow \frac{1}{\sqrt{2}} (|g\rangle + \lambda |k\rangle). \tag{17}$$

In this case, the photon will never exit the system so as to be counted by a detector at beam h . But when the absorber A is present, the wave function will evolve so that

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{2}} (|a\rangle + \lambda |b\rangle) \rightarrow \frac{1}{\sqrt{6}} ((|d\rangle + |e\rangle + |f\rangle) + \lambda(|d\rangle + \lambda |e\rangle + \mu |f\rangle)) \\ &\rightarrow \frac{1}{\sqrt{6}} (-\mu |d\rangle - \lambda |e\rangle + 2 |A\rangle) \\ &\rightarrow \frac{1}{\sqrt{18}} (|g\rangle + \lambda |k\rangle - 2\mu |h\rangle) + \frac{2}{\sqrt{6}} |A\rangle. \end{aligned} \tag{18}$$

We see here that there is a greater probability of losing the original photon to the absorber, which will happen with of the time. But now, when the photon remains, it will hit the detector at h 67% of the time, which it never did when the absorber was absent. So the non-classical aspect of the experiment is enhanced to 67%, rather than 50%, when a tritter replaces the beam-splitter. With higher n -beam critters, this percentage would be enhanced even more. These experiments can also be improved by combining them with Zeno paradox devices, as was done for the beam-splitter [5].

How Three Particles Can Be Better Than Two

We have claimed that anything you can do with two particles will be more interesting with three particles. We will illustrate with the example of quantum teleportation [6]. Actually, teleportation is performed with a three-particle system, but one of these particles is the passive incident state, while the active transformation is carried out by the other two particles. We shall see how three active particles can lead to more interesting results.

First we review the case with two particles. The procedure is outlined in Fig. 7. One starts with a particle, which we will assume is spin 1/2, in an arbitrary state,

$$\psi_1 = \alpha \uparrow_1 + \beta \downarrow_1, \tag{19}$$

where the subscript represents particle 1, the original particle, which is to be teleported from A to B . Between them, a source emits particles 2 and 3 in an entangled state φ_{23}^1 (see

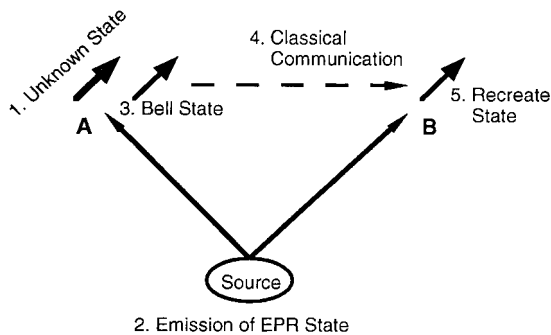


Fig. 7: Quantum Teleportation. The teleportation takes place through a five step process: (1) a particle in an unknown original state is located at A; (2) an entangled pair of particles is sent from a source to A and B; (3) a measurement is made at A to determine in which Bell state the pair of particles is located; (4) this information is communicated to B by classical means; (5) the experimenter at B then uses this information to manipulate his particle so that it ends up in the state of the original particle at A.

eq. (20)), and when particle 2 reaches A, a joint measurement is made between particles 1 and 2, to place them in a Bell state, φ_{12}^i . Here the subscript represents particles 1 and 2, while the superscript represents one of the four Bell states,

$$\begin{aligned}
 \varphi^1 &= \frac{1}{\sqrt{2}} (\uparrow\uparrow + \downarrow\downarrow), \\
 \varphi^2 &= \frac{1}{\sqrt{2}} (\uparrow\uparrow - \downarrow\downarrow), \\
 \varphi^3 &= \frac{1}{\sqrt{2}} (\uparrow\downarrow + \downarrow\uparrow), \\
 \varphi^4 &= \frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow).
 \end{aligned} \tag{20}$$

This measurement yields one of the Bell states. The amplitudes for the various outcomes are

$$\begin{aligned}
 \Psi_{\text{tot}} &= \psi_1 \varphi_{23}^1 = (\alpha \uparrow_1 + \beta \downarrow_1) \frac{1}{\sqrt{2}} (\uparrow_2 \uparrow_3 + \downarrow_2 \downarrow_3) \\
 &= \frac{1}{\sqrt{2}} (\alpha \uparrow_1 \uparrow_2 \uparrow_3 + \alpha \uparrow_1 \downarrow_2 \downarrow_3 + \beta \downarrow_1 \uparrow_2 \uparrow_3 + \beta \downarrow_1 \downarrow_2 \downarrow_3) \\
 &= \frac{1}{2} [\alpha(\varphi_{12}^1 + \varphi_{12}^2) \uparrow_3 + \alpha(\varphi_{12}^3 + \varphi_{12}^4) \downarrow_3 \\
 &\quad + \beta(\varphi_{12}^3 - \varphi_{12}^4) \uparrow_3 + \beta(\varphi_{12}^1 - \varphi_{12}^2) \downarrow_3] \\
 &= \frac{1}{2} [\varphi_{12}^1(\alpha \uparrow_3 + \beta \downarrow_3) + \varphi_{12}^2(\alpha \uparrow_3 - \beta \downarrow_3) \\
 &\quad + \varphi_{12}^3(\alpha \downarrow_3 + \beta \uparrow_3) + \varphi_{12}^4(\alpha \downarrow_3 - \beta \uparrow_3)] \\
 &= \frac{1}{2} [\varphi_{12}^1 R_1 \psi_3 + \varphi_{12}^2 R_2 \psi_3 + \varphi_{12}^3 R_3 \psi_3 + \varphi_{12}^4 R_4 \psi_3].
 \end{aligned} \tag{21}$$

As can be seen from eq. (21), the measurement that puts particles 1 and 2 in a Bell state also puts particle 3 into a state that is either the original state ψ , or a state that is connected

to it by some simple rotation and/or inversion. In eq. (21), we see that a measurement of the state φ^i reduces the state of particle 3 to this unique state. For example, if the measurement shows that particles 1 and 2 are in the state φ^3 , then the state of particle 3 differs from that of ψ by \uparrow and \downarrow having been interchanged, so that the transformation R_3 is just the inversion operator I .

The next stage of the teleportation is that the experimenter at B must be told the result of the Bell measurement at A . This step is done by a classical communication, perhaps over the telephone. (It is this step that prevents the teleportation from being carried out superluminally.) Once B knows the result of the measurement at A , he knows what transformation he must perform on particle 3 to convert it to the state ψ , namely the inverse transformation R_i^{-1} , which does not depend on either α or β . The end result is that the state ψ has been transferred from particle 1 to particle 3, without the particle having been sent, even though this state may be unknown to all parties involved.

We will now describe the same event in the case when there are three particles involved in the EPR source emission. In this case we take the emission state arbitrarily to be $\Phi_{234} = \frac{1}{\sqrt{2}} (\uparrow_2 \uparrow_3 \uparrow_4 + \downarrow_2 \downarrow_3 \downarrow_4)$. Then

$$\begin{aligned} \Psi_{\text{tot}} &= \psi_1 \Phi_{234} = \frac{1}{\sqrt{2}} (\alpha \uparrow_1 + \beta \downarrow_1) (\uparrow_2 \uparrow_3 \uparrow_4 + \downarrow_2 \downarrow_3 \downarrow_4) \\ &= \frac{1}{2} [\varphi_{12}^1 (\alpha \uparrow_3 \uparrow_4 + \beta \downarrow_3 \downarrow_4) + \varphi_{12}^2 (\alpha \uparrow_3 \uparrow_4 - \beta \downarrow_3 \downarrow_4) \\ &\quad + \varphi_{12}^3 (\alpha \downarrow_3 \downarrow_4 + \beta \uparrow_3 \uparrow_4) + \varphi_{12}^4 (\alpha \downarrow_3 \downarrow_4 - \beta \uparrow_3 \uparrow_4)]. \end{aligned} \tag{22}$$

Now, when the Bell state for particles 1 and 2 is measured, one can use the result to transport the state ψ to *either* particle 3 or particle 4. This is the additional feature introduced by the extra particle. One has the choice of where to teleport the result.

We shall demonstrate how this feature is exploited. Say one wants to teleport the result to particle 4. Then one measures particle 3 along the x -direction. The spin up and down states along the z -direction can be expressed in terms of those along the x -direction by

$$\begin{aligned} \uparrow &= \frac{1}{\sqrt{2}} (\leftarrow + \rightarrow), \\ \downarrow &= \frac{1}{\sqrt{2}} (\leftarrow - \rightarrow), \end{aligned} \tag{23}$$

where \leftarrow represents the state $|+x\rangle$, and \rightarrow represents the state $|-x\rangle$. Then, if the result of the Bell measurement on particles 1 and 2 was φ_{12}^1 , one has

$$\begin{aligned} &\frac{1}{2\sqrt{2}} \varphi_{12}^1 (\alpha (\leftarrow_3 + \rightarrow_3) \uparrow_4 + \beta (\leftarrow_3 - \rightarrow_3) \downarrow_4) \\ &= \frac{1}{2\sqrt{2}} \varphi_{12}^1 (\leftarrow_3 (\alpha \uparrow_4 + \beta \downarrow_4) + \rightarrow_3 (\alpha \uparrow_4 - \beta \downarrow_4)) \\ &= \frac{1}{2\sqrt{2}} \varphi_{12}^1 (\leftarrow_3 R_{1-} \psi_4 + \rightarrow_3 R_{1-} \psi_4). \end{aligned} \tag{24}$$

So now both the experimenters at particles 2 and 3 must communicate with the one at 4 by a classical transmission. Particle 2's says the Bell measurement yielded φ^1 , while parti-

cle 3's says that the x measurement yielded \leftarrow . Then the experimenter at 4 knows that he must apply the inverse transformation $(R_{1\leftarrow})^{-1}$ to his particle in order to obtain ψ for particle 4. Similarly, if one wanted to teleport to particle 3, one would have to measure along the x -direction at particle 4 and transmit the result to particle 3. The procedure is identical, because of the symmetry between the two particles.

We believe the results we have presented represent merely the first steps in using 3-particle entangled states, and that there are many fascinating effects there that have not been dreamed of yet. This work was supported in part by the NSF, grant #PHY-97-22614.

References

- [1] One of these proposals has recently succeeded, using an NMR system at Los Alamos: See R. LAFLAMME, E. KNILL, W. H. ZUREK, P. CATASTI and S. V. S. MARIAPPAN (in press).
- [2] D. M. GREENBERGER, M. A. HORNE, A. SHIMONY, and A. ZEILINGER, *Amer. J. Phys.* **58**, 1131 (1990).
- [3] M. RECK, A. ZEILINGER, H. BERNSTEIN, and P. BERTANI, *Phys. Rev. Lett.* **73**, 58 (1994).
- [4] AVSHALOM C. ELITZUR & LEV VAIDMAN, *Found. Physics* **23**, 987 (1993).
- [5] P. G. KWIAT, H. WEINFURTER, T. HERZOG, A. ZEILINGER, and M. A. KASEVICH, *Phys. Rev. Lett.* **74**, 4763 (1995).
- [6] C. H. BENNETT, G. BRASSARD, C. CRÉPEAU, R. JOZSA, A. PEREZ, and W. K. WOOTTERS, *Phys. Rev. Lett.* **70**, 1895 (1993).

(Manuscript received: January 1, 1999)