Physica D 50 (1991) 321–326
North-Holland

Zeno's paradox in quantum cellular automata

Gerhard Grössing[a] and Anton Zeilinger[b]
[a]*Atominstitut der Österreichischen Universitäten, Schüttelstrasse 115, A-1020 Vienna, Austria*
[b]*Institut für Experimentalphysik, Universität Innsbruck, Technikerstrasse 25, A-6020 Innsbruck, Austria*[1]

Received 17 November 1989
Accepted 25 November 1990
Communicated by H. Flaschka

The effect of Zeno's paradox in quantum theory is demonstrated with the aid of quantum mechanical cellular automata. It is shown that the degree of non-unitarity of the cellular automaton evolution and the frequency of consecutive measurements of cellular automaton states are operationally indistinguishable.

1. Zeno's paradox and quantum mechanical cellular automata

Among other qualities, a quantum cellular automaton (QCA) can be considered as a novel tool for the investigation of the coevolution of a large number of individual points on a lattice in some parameter space under the condition of quantum mechanical transition rules [1, 2].

One can therefore expect as a possible result of QCA research to obtain a new viewpoint of the evolution of quantum systems in general. Naturally, it will be of particular interest to study in the framework of QCA the relations between quantum systems and "macroscopic objects". For example, in recent publications we have investigated different strategies of introducing irreversibility into the quantum domain [3], and one of us has compared classical and quantum mechanical lattice properties with regard to their reversibility/irreversibility properties [4].

In the present paper we focus our interest on a particular aspect of the quantum mechanical measurement process, named "Zeno's paradox" [5] (or sometimes also the "watchdog effect" [6]), and we simulate this effect with the means of QCA.

What is the "Zeno's paradox" in quantum theory? In its simplest version [6] it describes the general behavior of a quantum system which is repeatedly measured in short time intervals such that its "motion is frozen", i.e. the internal dynamics of the system is totally suppressed.

Consider the state $|\psi\rangle$ of a quantum system with Hamiltonian H at time $t=0$. Now divide the time axis into small elements Δt. Then the "decay probability" $P(\Delta t)$ of *not* finding the state $|\psi\rangle$ at the time Δt is

$$\begin{aligned} P(\Delta t) &= 1 - |\langle\psi|\, e^{-iH\Delta t}\,|\psi\rangle|^2 \\ &\simeq \big(\langle\psi|H^2|\psi\rangle - (\langle\psi|H|\psi\rangle)^2\big)(\Delta t)^2 \\ &\quad + \mathcal{O}\big((\Delta t)^3\big). \end{aligned} \tag{1}$$

If one now repeats the measurement n times during Δt, the probability $P(\Delta t)$ reduces because of the quadratic time dependence to

$$P \to P' \simeq n\left(\frac{\Delta t}{n}\right)^2 \big(\langle\psi|H^2|\psi\rangle - (\langle\psi|H|\psi\rangle)^2\big). \tag{2}$$

[1]Address to which reprint requests should be sent.

Thus, in the limit $n \to \infty$ one obtains $P' = 0$. Consequently, the internal dynamics of the repeatedly measured quantum system is totally suppressed, a fact which leads to a disappearance of interferences between distinguished individual amplitudes and thus constitutes an irreversible process. Basically, this is a result of the non-unitary "collapse of the state vector" [6].

In order to simulate the Zeno's paradox with cellular automata (CA) we propose to study the "general quantum evolution" of a CA whose site is characterized by a complex amplitude $c(x,t)$, and whose evolution rule is

$$c(x,t+1) = c(x,t) - \mathrm{i}Ec(x,t) + \mathrm{i}\delta^* c(x+1,t) + \mathrm{i}\delta c(x-1,t). \tag{3}$$

Comparison of eq. (3) with refs. [1–4] shows that it is a more general evolution rule than the one used previously to define quantum cellular automata (QCA). For the latter, the constant E is chosen to vanish. Thus, for QCA the evolution rule is given by

$$c(x,t+1) = c(x,t) + \mathrm{i}\delta^* c(x+1,t) + \mathrm{i}\delta c(x-1,t). \tag{4}$$

The generalization of evolution rule (4) to the rule of eq. (3) is of particular interest, since for a special choice of the parameters eq. (3) corresponds to the discretized version of the Schrödinger equation [7]. That is, for real values of δ, i.e. for

$$\delta = \delta_c + \mathrm{i}\delta_c', \quad \varphi = \arctan(\delta_c'/\delta_c) = 0 \tag{5}$$

one obtains from (3) with $\delta = \frac{1}{2}E$:

$$c(x,t+1) = (1 - \mathrm{i}E)c(x,t) + \tfrac{1}{2}\mathrm{i}Ec(x+1,t) + \tfrac{1}{2}\mathrm{i}Ec(x-1,t). \tag{6}$$

Eq. (6) can now be interpreted with $V := 1 - E$ as the discretized Schrödinger equation for a particle with potential energy V. Comparings eqs. (4) and (6) one finds that for both evolution rules unitarity is approximately preserved as long as $\delta^2 \simeq 0$. This means that a QCA is comparable to a Schrödinger type evolution only for $\delta \to 0$. Nevertheless, as will be shown below, both evolution (4) and (6) exhibit the behavior due to Zeno's paradox in the simulation of "measurements" in cellular automata governed by quantum mechanical rules.

2. Measurements in cellular automata obeying the general quantum evolution rule

Denoting with $J = 1, \ldots, 120$ the sites at time step I of a cellular automaton, and attributing a complex number c_{IJ} to each site, the evolution rule (3) reads in a more convenient notation as

$$c_{I+1,J} = (1 - \mathrm{i}E)c_{IJ} + \mathrm{i}\delta c_{I,J-1} + \mathrm{i}\delta^* c_{I,J+1}. \tag{3'}$$

We shall present the CA in terms of probability maps attributing to each site (I,J) the real value $P_{IJ} = |c_{IJ}|^2$ with the normalization

$$\sum_J |c_{IJ}|^2 = 1. \tag{7}$$

Now we introduce a "measurement" at each site J in row $I = M$ (i.e. at "time" M), and we calculate the probability distribution in row $I = N$, with $N > M$. To do this, we carry out the following procedure for each amplitude c_{MJ}: First, choose $c_{MJ} = 1$, with $c_{IJ} = 0$ for $I \neq M$. Then calculate $c_{NJ'}$, with $J' \neq J$, with the help of eq. (3'). Introducing the notation

$$c_{NJ'J} := c_{NJ'}|_{c_{MJ=1}},$$

one obtains for each initial site J a set of amplitudes $c_{NJ'J}$ (see fig. 1). Finally, we calculate the

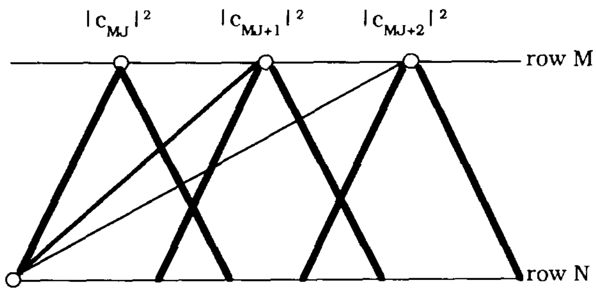


Fig. 1. Construction of the probability distribution $P_{NJ'} = |c_{NJ'}|^2$ at "time" $I = N$ after measurements have been performed at each site J at "time" $I = M$. The third index in $c_{NJ'N}$ indicates the contribution from the calculatory procedure described in the text following the choice of $c_{MJ} = 1$. (Very thick lines: $|c_{NJ'J}|^2$, thick lines: $|c_{NJ'J+1}|^2$, thin lines: $|c_{NJ'J+2}|^2$.)

sum

$$P_{NJ'} = \sum_J |c_{MJ}|^2 |c_{NJ'J}|^2 . \tag{8}$$

Thus, $P_{NJ'}$ is the value for the probability $|c_{NJ'}|^2$ in row N and column J' after a measurement has been performed in row M. The normalization condition now becomes for each time step $I > M$:

$$\sum_J |c_{IJ}|^2 |c_{NJ'J}|^2 = 1 . \tag{9}$$

3. Results

In the plates showing the probability maps for various quantum mechanical CA different colors represent different probabilities, and the number of pixels is 532×120 for each image. Plate I shows a typical result for a QCA (i.e. where $E = 0$ and δ is complex), whose evolution is altered at time step $I = 250$. From then on, continuous measurements are performed at each step $I \geq 250$ in the way outlined in section 2. The evolution runs from top to bottom, with one initial point at site $J = 60$, and $\delta = 0.5(1 + \mathrm{i})$. The transition from "ordinary CA evolution" to the sequence of continuous measurements starting at $I = 250$ is very clearly seen, the interference terms disappear leaving the "smeared out" distribution on the bottom part of plate I. However, the result does not quite show what one would expect from "Zeno's paradox", since the latter should manifest itself through constant values P_{NJ} for each J and for all states $N > M = 250$. The reason for this deviation from straight lines of the probability distribution in plate I, however, is obvious: The evolution for $\delta = 0.5(1 + \mathrm{i})$ is clearly non-unitary, and therefore the slight spreading of the probability distribution after time $I = 250$ accounts for this fact. So, let us choose a smaller δ and see what happens then!

Plate II plots a cellular automaton representing the discretized Schrödinger equation for $\delta = 0.05$ (i.e. $E = 0.1$). Here the effect due to Zeno's paradox can be clearly seen: after the onset of the continuous measurement at time $M = 250$, the probability distribution essentially remains constant for all later times (except for one new "stripe" at the edge of the figure which is due to the fact that the evolution is still not perfectly unitary for $\delta = 0.05$). Note also that the pattern in some regions is striped while in others it is not, a quality characteristic for the transitory regime between the striped QCA (see refs. [1, 2]) and the "smoother" behavior due to the discretized Schrödinger evolution. For illustration, plates III and IV show the two extremes of "QCA-type evolution" (plate III) and of "Schrödinger evolution" (plate IV).

In plate III, the parameters are chosen to be very small ($\delta = 0.01$ and $E = 0.02$) such that the Schrödinger evolution is practically indistinguishable from QCA evolution due to the smallness of E. Moreover, starting measurements at time $M = 250$, further measurements are performed not continuously but in intervals of $\Delta I = 20$. Whereas continuous measurement would produce a straight profile of the probability distribution, one now observes a slight spreading thereof.

As we have seen, such a spreading can be caused by the CA evolution being non-unitary.

Here, however, we observe another cause: the relative frequency of measurements different from continuous measurements. In principle, by just looking at the probability maps and the spreading of the profiles, one cannot distinguish between the two causes.

Moreover, there exists a curious feature of Schrödinger-type evolutions as the one presented in plate IV. There, the variables are $\delta = 0.25$ and $E = 0.5$. At time $M = 250$ continuous measurement sets in producing a spreading probability profile due to the non-unitarity of the evolution rule. Two points are particularly interesting: (1) The top part of plate IV, where no measurement is performed, looks the same for any Schrödinger evolution for which $\delta \gg 0$. Since the first term on the r.h.s. of eq. (6) becomes $(1 - \mathrm{i}E) \to -\mathrm{i}E$ for large E, one can divide the whole equation by E (altering just the normalization factor), such that the probability profile looks the same for any particular choice of E (or of δ, respectively). (2) However, one can still get some information out of such Schrödinger evolution if one starts at some time with continuous measurements: then, the rapidity of the spreading probability profile (i.e. the slope of its "edges") provides a useful measure for the non-unitarity of the CA evolution, i.e. one can obtain additional information on CA that looked the same as long as no measurements were performed. In fact, this property can also be found when applied to QCA (see plate I), i.e. it is a property of the "general quantum evolution" as described by eq. (3).

The dependence of the spreading probability profile (measured in terms of the slope k of its edges) on the choice of the frequency of measurements (presented in terms of time intervals ΔI) is given for various δ in fig. 2. The value of $k = 1/d$ is determined by measuring the distance d in time steps I from $I = M$ to the time when the probability profile reaches the boundary $J = 1$ and $J = 120$ of the CA.

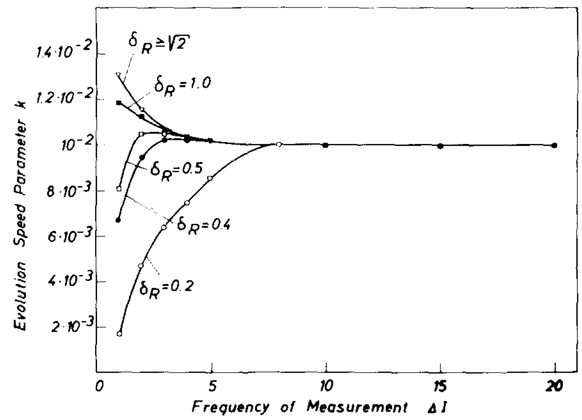


Fig. 2. Dependence of the spreading probability profile (measured in terms of the slope k of its edges) on the choice of the frequency of measurements (presented in terms of time intervals ΔI) for different choices of δ.

For very small δ and continuous measurement ($\Delta I = 1$), the slope k is practically zero (Zeno's paradox), and is growing towards a constant value for larger ΔI. The constant value for k is a limit reached for all choices of δ after sufficient long time intervals between successive measurements, which just means that in these cases the rare measurements have practically negligible influence on the CA evolution. It is reached for all δ when $\Delta I \geq 10$.

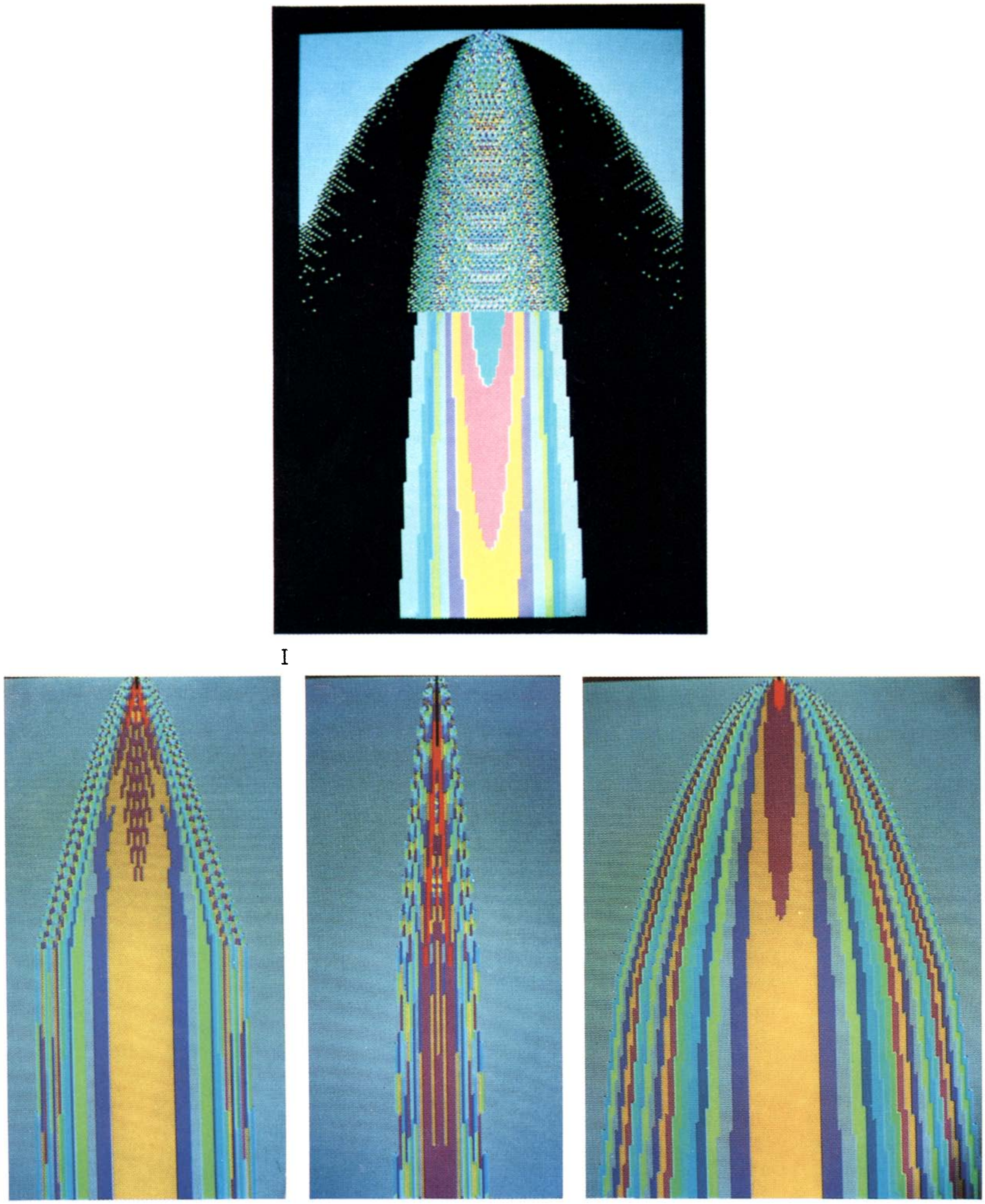
Plate I. Probability map for a quantum mechanical cellular automaton for $\delta = 0.5(1 + \mathrm{i})$ and one initial point at site $J = 60$. Continuous measurement sets in at "time" $I = 250$.

Plate II. Probability map for a cellular automaton representing the discretized Schrödinger equation for $\delta = 0.05$. Continuous measurement starting at $I = 250$ leads to the effect due to Zeno's paradox: the probability distribution essentially remains constant for all later times I.

Plate III. Probability map for a cellular automation representing the discretized Schrödinger equation for $\delta = 0.01$. Measurements, starting at $I = 250$, are repeated in intervals of $\Delta I = 20$.

Plate IV. Same as plate III, but with $\delta = 0.25$ and continuous measurement after $I = 250$.

In all four plates the "time axis" I runs from top to bottom.



I

II

III

IV

However, if δ is chosen to be large, the above behavior of growing slopes k is reversed. In fact, for all values $\delta \geq \sqrt{2}$, identical behavior is observed: the larger ΔI becomes, the smaller is the value for the slope k of the spreading probability profiles, finally merging into the constant value mentioned above. This is in accordance with the fact that $\delta \geq \sqrt{2}$ describes the QCA regime which can definitely be classified as non-unitary [1, 2].

Finally, we want to point out a consequence of this study of Zeno's paradox in quantum mechanical CA: the degree of non-unitarity and the frequency of consecutive measurements are operationally indistinguishable. This results from the finite resolution time of any possible series of "continuous measurements". In other words, a perfectly unitary evolution is an idealization which cannot be confirmed via measurement by any operational procedure. The probability maps of quantum mechanical cellular automata as presented here provide a particularly clear and direct demonstration of this quantum mechanical fact.

Acknowledgement

This work was supported by the Bundesministerium für Wissenschaft und Forschung under contract number Z1.19.153/3-26/85. Publication was made possible through a grant of the Österreichische Forschungsgemeinschaft.

References

[1] G. Grössing and A. Zeilinger, Quantum cellular automata, Complex Systems 2 (1988) 197, 611.

[2] G. Grössing and A. Zeilinger, A conservation law in quantum cellular automata, Physica D 31 (1988) 70.

[3] G. Grössing and A. Zeilinger, Structures in quantum cellular automata, Physica B 151 (1988) 366.

[4] G. Grössing, Comparison of classical and quantum lattice properties with the aid of cellular automata, Phys. Lett. A 131 (1988) 1.

[5] B. Misra and E.C.G. Sudarshan, The Zeno's paradox in quantum theory, J. Math. Phys. 18 (1977) 756.

[6] E. Joos, Continuous measurement: Watchdog effect versus golden rule, Phys. Rev. D 29 (1984) 1626.

[7] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman Lectures on Physics, part III, (Addison-Wesley, Reading, MA, 1965).