Quantum Cellular Automata

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Abstract. For cellular automaton machines getting increasingly smaller in size, a regime will be entered where quantum effects cannot be neglected. Ultimately, these quantum effects may very well be dominant. Quantum mechanically this fact is described by introducing probability amplitudes implying that one will not be able to know for certain whether the value at a given site is 0 or 1 at a given instant of time. We report results obtained by studying the evolution of one-dimensional cellular automata governed by quantum mechanical rules in such a way that superposition of probability amplitudes is permitted. We focus on strictly local interaction. The results are presented in the form of probability maps and clearly exhibit typical quantum features like constructive and destructive interference, beats and the like.

1. Introduction

Hitherto, cellular automata research was restricted to the study of deterministic or stochastic evolution [1]. In the present paper, we report on the first results extending cellular automata research into the region of quantum mechanics. This step, besides being of a fundamental interest, should also be of significance for the problem of understanding the impact of quantum physics on computer operation. This follows, because it has been shown [2] that certain classes of cellular automata are equivalent to Turing machines.

The main reason for quantum mechanics to enter computer operation some day is that, in order to become faster, these machines have to get

increasingly smaller in size. Therefore, a regime will be entered where quantum effects cannot be neglected and ultimately these effects may very well be dominant [3].

Consequently, one will not be able any more to know for certain whether the value at a given site is 0 or 1 at a given instant of time. Quantum mechanically this fact is described by introducing probability amplitudes.

We shall therefore attribute some complex number c_{IJ} to each cite of the cellular automaton and we shall construct transition rules in such a way that superposition of probability amplitudes is permitted. Studying the evolution of one-dimensional cellular automata, we focus on strictly local (i.e. nearest neighbor) interaction.

For small enough time steps, the unitary evolution operator U may be approximated using only the first-order term of its expansion

$$U = e^{-iHt/\hbar} \simeq 1 - iHt/\hbar.$$

Introducing periodical boundary conditions, the Hamiltonian becomes essentially for $|\delta| \ll 1$:

$$H = \begin{pmatrix} \ddots & \ddots & \ddots & & & 0 \\ & \delta^* & 0 & \delta & & \\ & & \delta^* & 0 & \delta & \\ & & & \delta^* & 0 & \delta \\ 0 & & & & \ddots & \ddots & \ddots \end{pmatrix} \tag{1.1}$$

Thus, the corresponding transition rule for quantum cellular automata specified in this way is

$$c_{I+1,J} = \frac{1}{\sqrt{N}}(c_{I,J} + i\delta c_{I,J-1} + i\delta^* c_{I,J+1}) \tag{1.2}$$

where I and J denote time-step and site location of the one-dimensional quantum cellular automaton respectively, and N is a normalization factor such that

$$\sum_J |c_{I,J}|^2 = 1 \text{ for all } I.$$

For a cellular automaton consisting of two sites only, this would correspond to the rule

$$\begin{pmatrix} a' \\ b' \end{pmatrix} =$$

$$\frac{1}{\sqrt{N}}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + i\delta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + i\delta^* \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right\}.$$

Clearly, (1.1) represents a unitary evolution for small values of δ only. For larger values of δ, other matrix elements farther off the diagonal would have to be nonzero in a very specific way to preserve unitarity. This would imply nonlocality. In contrast, we decided to study strictly local rules. These we define such that the amplitude at a given site and time depends only on this sites' amplitude and that of its nearest neighbors at the previous time step. In other words, we adopt equation (1.2) to be the rule governing the evolution of our quantum CAs independent of the size of δ. Rather than abandoning locality, we believe our choice to be a good candidate when envisaging possible future realizations of quantum cellular automata machines. Also, this approach provides a natural procedure for the transition between the quantum and the classical domain.

2. Results

The results are presented in the form of probability maps, i.e. we plot the "temporal" evolution of one-dimensional quantum cellular automata in terms of the normalized probability values $P_{IJ} = c^*_{IJ} c_{IJ}$ for each site J at each time step I. Different shades of grey represent different probabilities.

Our main interest in this paper is to study the evolution of quantum cellular automata as a function of the size of the off-diagonal elements δ in the Hamiltonian. That is, we want to investigate the dependence of the resulting patterns on the relative weighting of the nearest neighbor's contributions.

In the figures plotting the probability maps the number of pixels is 120×532 for one image, and 120×1596 for three consecutive images respectively. Generally, we vary the size of the off-diagonal contributions $\delta = \delta_c(1 + i)$ by varying δ_c and we vary the initial point configurations.

3. Probability maps with one initial site of nonzero amplitude

The most obvious feature common to quantum cellular automata created by rule (1.2) for all $\delta_c > \sqrt{2}$ is their striped pattern, i.e. sites with relatively high and relatively low intensities alternate regularly when the resulting pattern is observed at a specific time step I. To see how this comes about, one has to consider the explicit time evolution of the quantum cellular automaton. Starting with one nonzero initial point $c_{IJ} \neq 0$ one obtains the probability values for the "time" $I + 1$.

$$|c_{I+1,J}|^2 = \frac{1}{N}, |c_{I+1,J-1}|^2 = |c_{I+1,J+1}|^2 = \frac{2}{N}\delta_c^2 \tag{3.1}$$

and for "time" $I + 2$

$$|c_{I+2,J}|^2 = \frac{2}{N'}|\frac{1}{2} - 2\delta_c^2|^2 \tag{3.2}$$

$$|c_{I+2,J\pm1}|^2 = \frac{4}{N'}\delta_c^2$$

$$|c_{I+2,J\pm2}|^2 = \frac{2}{N'}\delta_c^4$$

Thus, one obtains for $\delta_c > \sqrt{2}$:

$$|c_{I+2,J}|^2 > |c_{I+2,J\pm2}|^2 > |c_{I+2,J\pm1}|^2 \tag{3.3}$$

which corresponds to an overall striped pattern of the probability maps for all quantum cellular automata governed by rule (1.2) with $\delta_c > \sqrt{2}$. For $\delta_c \leq \sqrt{2}$ striped patterns may arise, too, but they will in general occur only locally.

To show characteristic results, figures 1 through 3 present examples with one initial site of nonzero amplitude. In figure 1 we plot a quantum cellular automaton with $\delta_c = 0.02$. The resulting pattern exhibits a striped wave-like structure with interferences around the edges. For comparison, figures 2a through c show a quantum cellular automaton with $\delta_c = 20$ and one initial point. The ellipses typical for this range of δ_c gradually flatten and eventually form "plane wave surfaces." Finally, figure 3 shows a quantum cellular automaton with $\delta_c = 4000$ and one initial point. Increasing the value of δ_c does not change the pattern. One can therefore speak of a "final state" pattern.

4. Probability maps with more than one initial site of nonzero amplitude

For the patterns studied by us so far with more than one initial site of nonzero amplitude, two characteristic statements can be made.

1. As a consequence of patterns being striped in the way described above one can formulate a *relative initial point location rule*: Whenever the number of intermediate states in one row of equal time I between two initial points is odd, one obtains symmetrical or melting features (e.g. melting ellipses). Whenever that number is even, the figures are distorted or die out (the latter being the case for small-grained patterns). Generally, if the number of initial points is increased one can obtain more and more complex behavior (dying out of some figures and melting of others, etc.) which leads to an increasing sensitivity to the intial conditions—changing one out of, say, four initial points by moving it one site to the right or left (or by reducing or increasing the value of its amplitude) can lead to dramatic differences in the resulting patterns.

2. There is one specific condition that—if fulfilled—produces a stable pattern after a few time steps. This one may be called the *constancy in time criterion*: Whenever the initial configuration is such that the location of the (at least two) sets of initial points, with each set containing at least one initial point, is rotation symmetric along the axis of the torus defined by the period boundary conditions, one obtains stable patterns after a few initial time steps (i.e. typically between 300 and 500) which are then conserved for all later times.
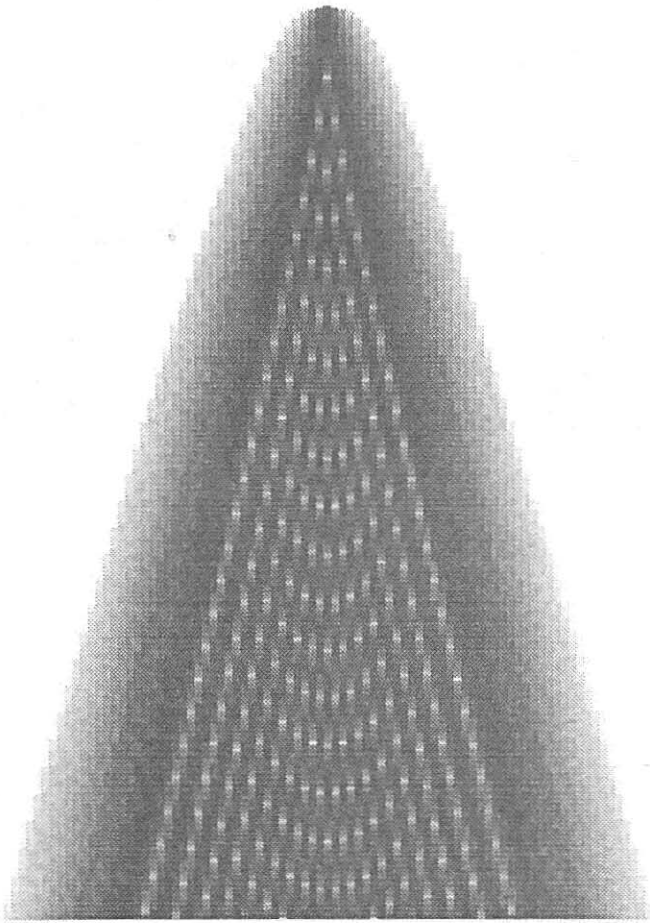
Figure 1: Quantum cellular automaton with $\delta_c = 0.02$ and one initial point. The resulting pattern exhibits a striped wave-like structure with interferences around the edges.
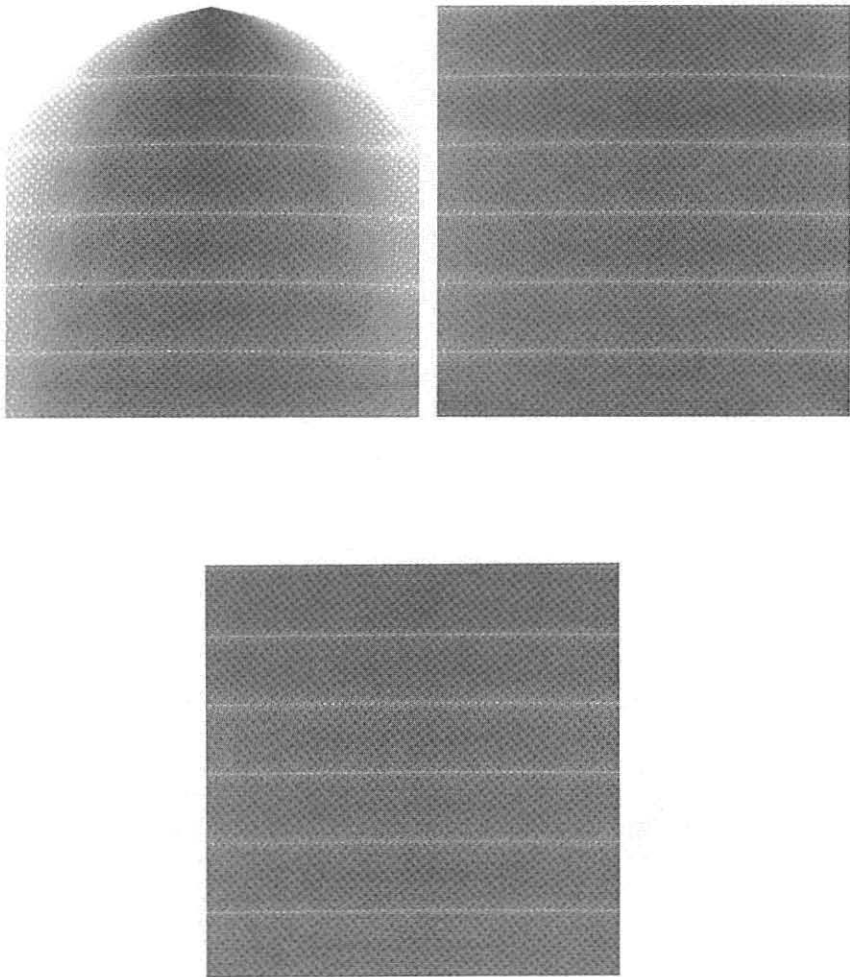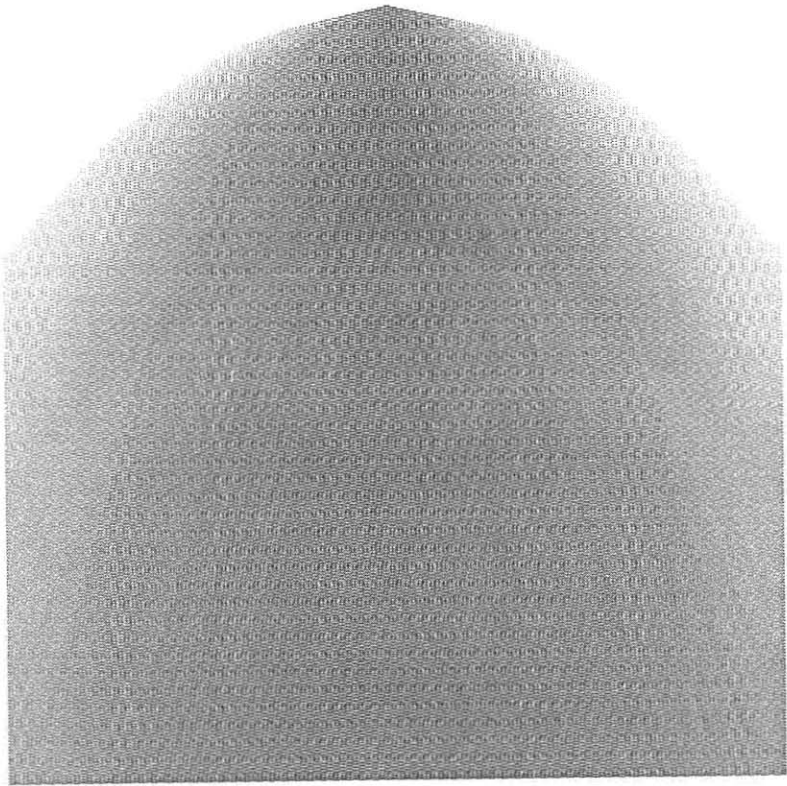
Figure 2: Quantum cellular automaton with $\delta_c = 20$ and one initial point. The ellipses typical for this range of δ_c gradually flatten with time and eventually form "plane wave surfaces."

Figure 3: Quantum cellular automaton with $\delta_c = 4000$ and one initial point. Increasing the value of δ_c does not change the pattern. One can therefore speak of a "final state" pattern.

The following figures present examples of the various consequences result-
ing from the features described above. Figure 4 shows a quantum cellular
automaton with $\delta_c = 0.08$ and two initial points at locations $(I, J) = (1, 30)$
and $(1, 90)$. The two initial amplitudes are chosen with slightly different
values (1 and 0.9), and the resulting pattern shows corresponding slight dif-
ferences in the intensity distribution. Note that the two evolutions never
merge. Figure 5 presents a quantum cellular automaton with $\delta_c = 0.2$ and
two equal initial amplitudes at $(I, J) = (1, 40)$ and $(1, 80)$. Note that here
the two evolutions merge to form one connected pattern. Figure 6 shows a
quantum cellular automaton with $\delta_c = 0.5$ and four initial points with equal
amplitudes. The initial points are located rotationally invariant with respect
to the axis of the torus generated by the periodic boundary conditions. Con-
sequently, the resulting pattern stabilizes after approximately 500 time steps
and remains stable for all later times. In figure 7 we plot a quantum cel-
lular automaton with $\delta_c = 10$ and six equal initial amplitudes at locations
$J = 15$, 30, 45, 75, 90, and 105. The pattern stabilizes after a few initial
time steps. Finally, figures 8a through c show a quantum cellular automaton
with $\delta_c = 50$ and four initial points at $J = 1$, 40, 43, and 80, with equal
amplitudes providing a pattern that is irregular in the beginning and then
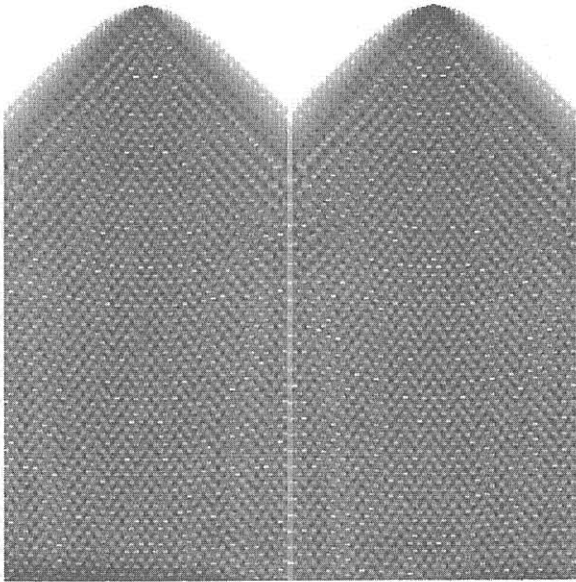gradually becomes more regular.

Figure 4: Quantum cellular automaton with $\delta_c = 0.08$ and two initial
points at locations $(I, J) = (1,30)$ and $(1,90)$. The two initial am-
plitudes are chosen with slightly different values (1 and 0.9), and the
resulting pattern shows corresponding slight differences in the inten-
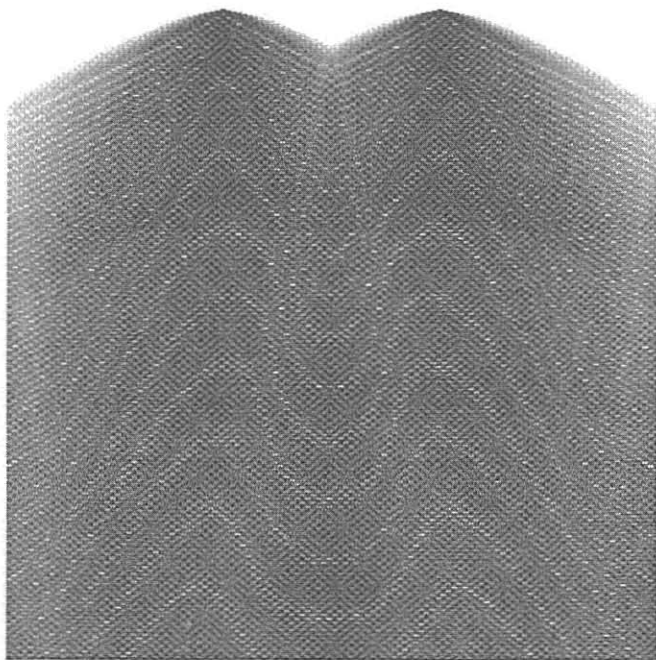sity distribution. Note that the two evolutions never merge.

Figure 5: Quantum cellular automaton with $\delta_c = 0.2$ and two equal initial amplitudes at $(I, J) = (1,40)$ and $(1,80)$. Note that here the two evolutions merge to form one connected pattern.
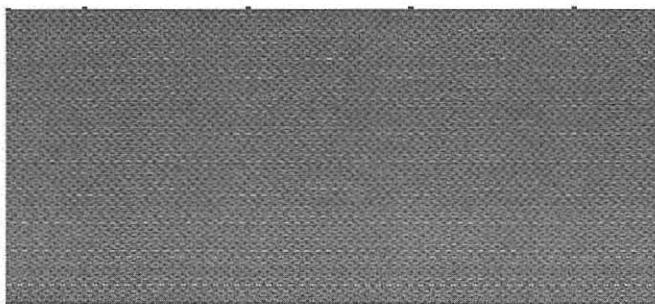

Figure 6: Quantum cellular automaton with $\delta_c = 0.5$ and four initial points with equal amplitudes. The initial points are located rotationally invariant with respect to the axis of the torus generated by the periodic boundary conditions. Consequently, the resulting pattern stabilizes after approximately 500 time steps and remains stable for all later times.
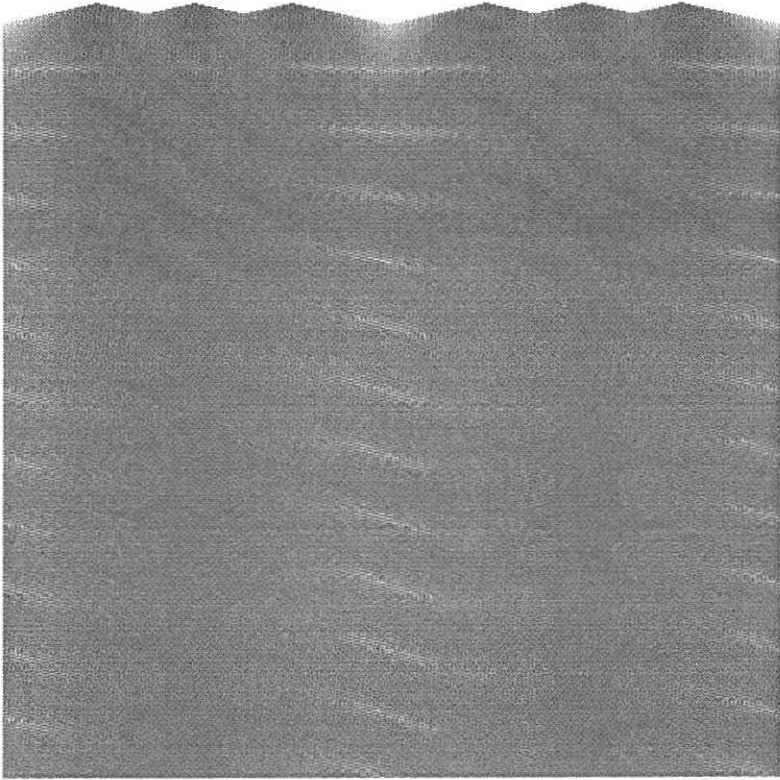
Figure 7: Quantum cellular automaton with $\delta_c = 10$ and six equal initial amplitudes at locations $J = 15$, 30, 45, 75, 90, and 105. The pattern stabilizes after a few initial time steps.
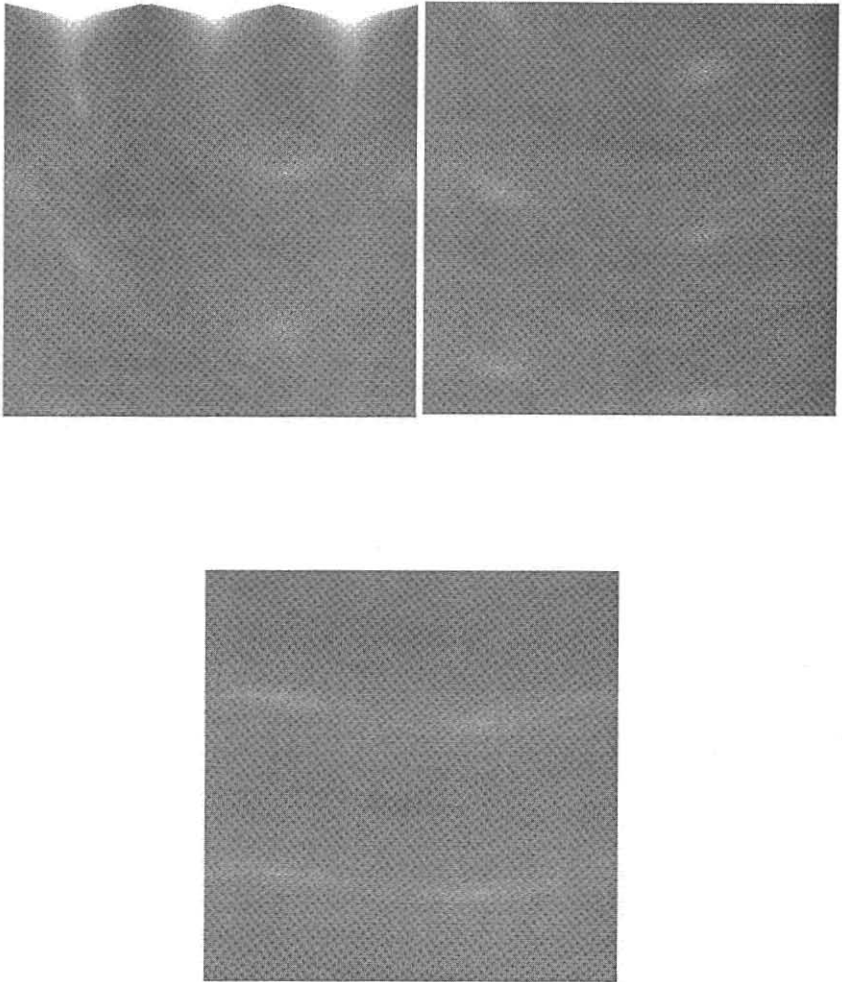
Figure 8: Quantum cellular automaton with $\delta_c = 50$ and four initial points at $J = 1$, 40, 43, and 80, proividing an irregular pattern that gradually becomes more regular.

The examples presented above are not arbitrary but are chosen as representations of various classes we found upon variation of the parameter δ_c. The classes can be characterized by their patterns and "prototype-values" of δ_c as follows: striped waves ($\delta_c = 0.02$), separated interference patterns ($\delta_c = 0.08$), melting interference patterns ($\delta_c = 0.2$), ripples ($\delta_c = 0.5$), elliptical disks ($\delta_c = 20$), and "final state" disk ($\delta_c \gtrsim 4000$). The resulting maps clearly exhibit typical quantum features such as constructive and destructive interferences, beats, and the like.

Acknowledgements

Note

Since the first presentation of this work at the 1986 MIT conference on Cellular Automata, we have continuously studied properties of quantum cellular automata. Among the published results we mention papers discussing irreversibility [4] and a conservation law [5] in QCAs.

References

[1] Stephen Wolfram, "Statistical Mechanics of Cellular Automata," *Review of Modern Physics*, **55** (1983) 642.

[2] J. von Neumann, *Theory of self-reproducing automata*, edited by A. W. Burks (University of Illinois Press, 1966).

[3] R. P. Feynman, "Simulating Physics with Computers," *Int. Journal of Theoretical Physics*, **21:6,7** (1982) 467.

[4] G. Grössing and A. Zeilinger, "Structures in Quantum Cellular Automata," *Physica B*, in press.

[5] G. Grössing and A. Zeilinger, "A Conservation Law in Quantum Cellular Automata," *Physica D*, in press.