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General Formulation of Spin Rotations in Neutron Interferometry

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Starting from the fact, that different rotations can lead to states with the same polarization but different phases the expressions of intensity and polarization of the forward diffracted beam after a Laue-case single crystal neutron interferometer are derived for the general case of phase shifts and spin rotations in both beam paths of the interferometer. A basic and instructive experiment where the polarization of the emerging beam is rotated by phase shift is discussed.

1. Introduction

The successful development of neutron interferometers [1, 2] based on X-ray interferometers using Laue-diffraction [3, 4] makes it possible to do some new kinds of experiments. In a neutron interferometer of that type, an incoming neutron beam is split into two coherent partial beams which are widely separated. The two beams leaving the interferometer represent a coherent superposition of these two partial beams. Measurements with unpolarized incident neutrons have already been reported on nuclear phase shift [1, 2] gravitational phase shift [5] and simple magnetic spin rotation [6, 7] including some effects of simultaneous phase shift and spin rotation [8]. Furthermore, the equivalence of rotations by homogeneous magnetic fields and helical fields was demonstrated theoretically [9].

The aim of the present paper is to develop the formalism of the general case of different phase shifts and spin rotations in the two partial beams. Special attention will be paid to the behaviour of the polarization vector of the neutron. As discussed previously by Rekveldt [10] and Mezei [11], the often used simple approach of considering the neutron beam as being composed of two oppositely polarized beams which are mutually incoherent provides incomplete information. In some experiments, additional information can be gained when not only the intensity of the beam and the z-component of its polarization are measured but also the x- and y-components as well. This also holds for other experiments discussed in this paper.

In Section 2 some general considerations are presented showing that different operators can lead to final states with the same polarization but different phases; these phase differences have no physical consequences in conventional experiments but can be measured by neutron interferometry. The expressions of intensity and polarization of the emerging forward diffracted neutron beam are derived in Section 3 for the general case mentioned above; in Section 4 a basic and instructive example, which can easily be verified experimentally, is discussed.

2. Basic Considerations

In general, any spinor can be written in the form

$$\psi = \begin{pmatrix} a\, e^{i\chi} \\ b\, e^{i\xi} \end{pmatrix} \tag{2.1}$$

with a, b, χ, and ξ being real and with the normalization condition $a^2 + b^2 = 1$. The polarization of a neutron beam described by (2.1) is

$$\mathbf{P} = \psi^+ \boldsymbol{\sigma} \psi = (2ab\cos(\xi - \chi), 2ab\sin(\xi - \chi), a^2 - b^2). \tag{2.2}$$

As an example of operations which can be performed on the spinor (2.1) we consider a 180° rotation around the x-axis, which can be represented by

$$U = e^{i\varphi} \sigma_x. \tag{2.3}$$

This leads to the spinor

$$\psi' = U\psi = e^{i\varphi}\begin{pmatrix} b\,e^{i\xi} \\ a\,e^{i\chi} \end{pmatrix}. \tag{2.4}$$

It is usually stated, that the phase φ may be arbitrarily chosen and has no effect on the physical properties of the system [12]. In particular, it does not affect the transformation law for observables; so, as an example, the polarization of the spinor (2.4) is

$$\mathbf{P}' = (2ab\cos(\xi-\chi),\ -2ab\sin(\xi-\chi),\ -(a^2-b^2)). \tag{2.5}$$

As expected, the phase factor cancels out. But if we now consider an interference experiment where the rotation operation (2.3) is performed only on one half of the spinor (2.1) we obtain the final spinor

$$\begin{aligned}\psi' &= \tfrac{1}{2}\psi + \tfrac{1}{2}e^{i\varphi}\sigma_x\psi \\ &= \tfrac{1}{2}\begin{pmatrix} a\,e^{i\chi} + b\,e^{i(\varphi+\xi)} \\ b\,e^{i\xi} + a\,e^{i(\varphi+\chi)} \end{pmatrix}.\end{aligned} \tag{2.6}$$

Evidently, the observables of (2.6) now depend on the particular choice of the phase φ of the operator (2.3). The general analysis of this fact and of some consequences is given in the next section.

3. General Formulation

As is well known, time dependence in quantum mechanics can be described by a unitary evolution operator

$$\psi(t) = U\psi(t_0). \tag{3.1}$$

If the Hamiltonian H of the system does not depend explicitely upon time this operator is

$$U = \exp[-(i/\hbar)H(t-t_0)]. \tag{3.2}$$

In the following we restrict ourselves to unitary operators which do not change the momentum of the neutrons. This is a physically plausible restriction because the very narrow reflection curve of the ideal crystal neutron interferometer [13] allows only that neutrons with very small changes of the momentum to influence the interference pattern. This means that neutron scattering by small angles such as seconds of arc or with very small momentum changes are disregarded. Under the assumptions made above, the unitary operator can be written as

$$U = e^{i\chi}(\alpha_0 + i\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}) \tag{3.3}$$

where χ is a phase change and α_0 and $\boldsymbol{\alpha}$ are associated with a general spinor rotation [14]. The condition that (3.3) is a unitary operator implies that $\alpha_0^2 + \boldsymbol{\alpha}^2 = 1$. The phase factor $\exp(i\chi)$ is usually viewed as inconsequential and ignored [14] but in our case, in contrast, it will lead to measurable consequences.

In the case of the ideal and focused interferometer [13] without phase shifters and fields, which we call the "initial state" i, we have for the forward diffracted beam

$$\psi_i = \psi_{i\mathrm{I}} + \psi_{i\mathrm{II}} \tag{3.4a}$$

and

$$\psi_{i\mathrm{I}} = \psi_{i\mathrm{II}}. \tag{3.4b}$$

Here $\psi_{i\mathrm{I}}$ and $\psi_{i\mathrm{II}}$ denote those parts of the forward diffracted beam associated with beam paths I and II respectively. Evidently, this does not mean a localization of the neutron in one of the beam paths which would destroy the interference pattern [14, 15]. We now assume that the waves within the interferometer are modified in a way which can be described by unitary operators of the general form (3.3)

$$\begin{aligned} U_{\mathrm{I}} &= e^{i\chi}(\alpha_0 + i\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}), \\ \alpha_0^2 &+ \boldsymbol{\alpha}^2 = 1, \end{aligned} \tag{3.5a}$$

and

$$\begin{aligned} U_{\mathrm{II}} &= e^{i\xi}(\beta_0 + i\boldsymbol{\beta}\cdot\boldsymbol{\sigma}), \\ \beta_0^2 &+ \boldsymbol{\beta}^2 = 1. \end{aligned} \tag{3.5b}$$

In the "final state", as defined here, the wave function of the forward diffracted beam is

$$\psi_f = U_{\mathrm{I}}\psi_{i\mathrm{I}} + U_{\mathrm{II}}\psi_{i\mathrm{II}} = M\psi_i, \tag{3.6a}$$

with

$$M = e^{i\xi}(\gamma_0 + i\boldsymbol{\gamma}\cdot\boldsymbol{\sigma})/2, \tag{3.6b}$$

$$\gamma_j = e^{i\mu}\alpha_j + \beta_j, \tag{3.6c}$$

where $\mu = \xi - \chi$ is the net relative phase change between the beam paths I and II. In general the operator M is not unitary.

3.1. Final Intensity

The intensity of the final state is obtained by evaluating

$$I_f = \psi_f^+\psi_f = \psi_i^+ M^+ M\psi_i \tag{3.7}$$

where

$$M^+M = (\gamma_0^* - i(\boldsymbol{\gamma}\cdot\boldsymbol{\sigma})^+)e^{-i\xi}e^{i\xi}(\gamma_0 + i\boldsymbol{\gamma}\cdot\boldsymbol{\sigma})/4. \tag{3.8}$$

Defining $\boldsymbol{\gamma}^*$ as the complex conjugate of $\boldsymbol{\gamma}$ we get

$$\begin{aligned} M^+M &= [\gamma_0\gamma_0 + i(\gamma_0^*\boldsymbol{\gamma} - \gamma_0\boldsymbol{\gamma}^*)\cdot\boldsymbol{\sigma} + (\boldsymbol{\gamma}^*\cdot\boldsymbol{\sigma})(\boldsymbol{\gamma}\cdot\boldsymbol{\sigma})]/4 \\ &= [\gamma_0^*\gamma_0 + i(\gamma_0^*\boldsymbol{\gamma} - \gamma_0\boldsymbol{\gamma}^*)\cdot\boldsymbol{\sigma} + \boldsymbol{\gamma}^*\cdot\boldsymbol{\gamma} + i(\boldsymbol{\gamma}^*\times\boldsymbol{\gamma})\cdot\boldsymbol{\sigma}]/4. \end{aligned} \tag{3.9}$$

Here we used the identity

$$(\mathbf{A}\cdot\boldsymbol{\sigma})(\mathbf{B}\cdot\boldsymbol{\sigma})=\mathbf{A}\cdot\mathbf{B}+i\boldsymbol{\sigma}(\mathbf{A}\times\mathbf{B}) \tag{3.10}$$

for $\mathbf{A}$ and $\mathbf{B}$ commuting with $\boldsymbol{\sigma}$. By use of the definition (3.6c) we can write

$$\begin{aligned}&\gamma_j^*\gamma_j=\alpha_j^2+\beta_j^2+2\alpha_j\beta_j\cos\mu,\\&i(\gamma_0^*\boldsymbol{\gamma}-\gamma_0\boldsymbol{\gamma}^*)=2(\alpha_0\boldsymbol{\beta}-\beta_0\boldsymbol{\alpha})\sin\mu,\\&i(\boldsymbol{\gamma}^*\times\boldsymbol{\gamma})=2\boldsymbol{\alpha}\times\boldsymbol{\beta}\sin\mu,\end{aligned} \tag{3.11}$$

which leads to

$$\begin{aligned}&M^+M=[1+(\alpha_0\beta_0+\boldsymbol{\alpha}\cdot\boldsymbol{\beta})\cos\mu\\&+\sin\mu(\alpha_0\boldsymbol{\beta}-\beta_0\boldsymbol{\alpha}+\boldsymbol{\alpha}\times\boldsymbol{\beta})\boldsymbol{\sigma}]/2.\end{aligned} \tag{3.12}$$

Now we introduce the intensity and the polarization of the initial state as

$$\begin{aligned}&I_i=\psi_i^+\psi_i,\\&\mathbf{P}_i=\psi_i^+\boldsymbol{\sigma}\psi_i/\psi_i^+\psi_i,\end{aligned} \tag{3.13}$$

and obtain for the intensity of the final state

$$\begin{aligned}&I_f=\frac{I_i}{2}[1+(\alpha_0\beta_0+\boldsymbol{\alpha}\cdot\boldsymbol{\beta})\cos\mu\\&+(\alpha_0\boldsymbol{\beta}-\beta_0\boldsymbol{\alpha}+\boldsymbol{\alpha}\times\boldsymbol{\beta})\mathbf{P}_i\sin\mu].\end{aligned} \tag{3.14}$$

3.2. Final Polarization

The polarization of the final state is defined as

$$\mathbf{P}_f=\psi_f^+\boldsymbol{\sigma}\psi_f/\psi_f^+\psi_f=\psi_i^+M^+\boldsymbol{\sigma}M\psi_i/I_f. \tag{3.15}$$

Thus, we have to evaluate

$$\begin{aligned}M^+\boldsymbol{\sigma}M&=[\gamma_0^*-i(\boldsymbol{\gamma}\cdot\boldsymbol{\sigma})^+]\,\boldsymbol{\sigma}[\gamma_0+i(\boldsymbol{\gamma}\cdot\boldsymbol{\sigma})]/4\\&=[\gamma_0^*\gamma_0\boldsymbol{\sigma}+i\gamma_0^*\boldsymbol{\sigma}(\boldsymbol{\gamma}\cdot\boldsymbol{\sigma})-i\gamma_0(\boldsymbol{\gamma}^*\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}+(\boldsymbol{\gamma}^*\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}(\boldsymbol{\gamma}\cdot\boldsymbol{\sigma})]/4.\end{aligned} \tag{3.16}$$

Using now the identies

$$\begin{aligned}&(\boldsymbol{\sigma}\cdot\mathbf{A})\boldsymbol{\sigma}=\mathbf{A}+i(\boldsymbol{\sigma}\times\mathbf{A}),\\&\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\mathbf{A})=\mathbf{A}-i(\boldsymbol{\sigma}\times\mathbf{A}),\end{aligned} \tag{3.17}$$

we get

$$\begin{aligned}&M^+\boldsymbol{\sigma}M=[(\gamma_0^*\gamma_0-\boldsymbol{\gamma}^*\cdot\boldsymbol{\gamma})\boldsymbol{\sigma}+i(\gamma_0^*\boldsymbol{\gamma}-\gamma_0\boldsymbol{\gamma}^*)+\gamma_0^*(\boldsymbol{\sigma}\times\boldsymbol{\gamma})\\&+\gamma_0(\boldsymbol{\sigma}\times\boldsymbol{\gamma}^*)+(\boldsymbol{\gamma}^*\cdot\boldsymbol{\sigma})\boldsymbol{\gamma}+\boldsymbol{\gamma}^*(\boldsymbol{\gamma}\cdot\boldsymbol{\sigma})-i\boldsymbol{\gamma}^*\times\boldsymbol{\gamma}]/4.\end{aligned} \tag{3.18}$$

With the identities (3.11) and

$$\begin{aligned}&\gamma_0^*(\boldsymbol{\sigma}\times\boldsymbol{\gamma})+\gamma_0(\boldsymbol{\sigma}\times\boldsymbol{\gamma}^*)\\&=2\boldsymbol{\sigma}\times[\alpha_0\boldsymbol{\alpha}+\beta_0\boldsymbol{\beta}+(\alpha_0\boldsymbol{\beta}+\beta_0\boldsymbol{\alpha})\cos\mu],\\&(\boldsymbol{\gamma}^*\cdot\boldsymbol{\sigma})\boldsymbol{\gamma}+(\boldsymbol{\gamma}\cdot\boldsymbol{\sigma})\boldsymbol{\gamma}^*\\&=2[(\boldsymbol{\alpha}+\boldsymbol{\beta}\cos\mu)\cdot\boldsymbol{\sigma}]\boldsymbol{\alpha}+2[(\boldsymbol{\alpha}\cos\mu+\boldsymbol{\beta})\cdot\boldsymbol{\sigma}]\boldsymbol{\beta},\\&i(\boldsymbol{\gamma}^*\times\boldsymbol{\gamma})=2\boldsymbol{\alpha}\times\boldsymbol{\beta}\sin\mu,\\&\gamma_0^*\gamma_0-\boldsymbol{\gamma}^*\cdot\boldsymbol{\gamma}=2\alpha_0^2+2\beta_0^2-2+2\alpha_0\beta_0\cos\mu-2\boldsymbol{\alpha}\cdot\boldsymbol{\beta}\cos\mu\end{aligned} \tag{3.19}$$

and by the use of (3.13) we finally obtain

$$\begin{aligned}\mathbf{P}_f=\frac{I_i}{2I_f}\{&[\alpha_0^2+\beta_0^2-1+(\alpha_0\beta_0-\boldsymbol{\alpha}\cdot\boldsymbol{\beta})\cos\mu]\mathbf{P}_i\\&-\boldsymbol{\alpha}\times\boldsymbol{\beta}\sin\mu+[(\boldsymbol{\alpha}+\boldsymbol{\beta}\cos\mu)\mathbf{P}_i-\beta_0\sin\mu]\boldsymbol{\alpha}\\&+[(\boldsymbol{\alpha}\cos\mu+\boldsymbol{\beta})\mathbf{P}_i+\alpha_0\sin\mu]\boldsymbol{\beta}\\&-[\alpha_0\boldsymbol{\alpha}+\beta_0\boldsymbol{\beta}+(\alpha_0\boldsymbol{\beta}+\beta_0\boldsymbol{\alpha})\cos\mu]\times\mathbf{P}_i\}.\end{aligned} \tag{3.20}$$

4. Discussion

The expressions of the emerging intensity (3.14) and the polarization (3.20) of the forward diffracted beam are the general solutions if we exclude or disregard changes of the neutron momentum. This formulae hold for an idealized neutron interferometer for which the coherence properties of the two partial beams are sufficiently retained so that full modulation of the interference pattern can be achieved. In reality, deviations occur because of effects like imperfections of the interferometer, neutron wavelength dispersion, and beam path inhomogeneities [13]. These effects diminish the amplitudes of the intensity oscillations as well as the polarization but the functional dependence is basically similar. The corresponding values of the intensity of the emerging deviated beam are readily obtained using conservation laws.

The general formulae, (3.14) and (3.20), can now be used to calculate and to predict the results of experiments in which the neutron waves in one or both beam paths are modified. To do this, it is only necessary, to find the appropriate unitary operators, (3.2), and to represent them in the required form (3.5). The analytical results derived here can be applied to advantage in experiments such as simultaneous nuclear phase shift and magnetic spin rotation [8]. This application is particularly straightforward for the case of rotations presented in the Section 2; this, incidently leads to different intensities and polarizations.

As an example of the application of this formalism, we now consider the case of a rotation around an arbitrary axis. Here the unitary operator is

$$U=e^{-i\boldsymbol{\sigma}\cdot\boldsymbol{\varphi}/2}=\cos(\varphi/2)-i\boldsymbol{\sigma}\cdot\mathbf{e}\sin(\varphi/2), \tag{4.1}$$

where $\boldsymbol{\varphi}$ is the rotation vector describing a rotation by the angle φ around the direction of the unit vector $\mathbf{e}=\boldsymbol{\varphi}/\varphi$. In that case we obtain, for the coefficients in the representation (3.5),

$$\alpha_0=\cos(\varphi/2),\qquad \boldsymbol{\alpha}=-\mathbf{e}\sin(\varphi/2). \tag{4.2}$$

We note that the 2π-rotations can be observed directly when these equations, (4.2), are inserted into the expressions of intensity and polarization, (3.14) and (3.20).

As a more specialized case, we now will examine a basic and instructive experiment with polarized incident neutrons. We assume, that the neutron spin is rotated in one beam path by an angle of 180° around an axis orthogonal to the initial polarization direction and afterwards the phase is modified in one partial beam. The beam leaving the interferometer is composed then of two oppositely polarized neutron beams. Here we can use the unitary operators

$$U_{\mathrm{I}}=e^{i\chi}\,e^{-i\boldsymbol{\sigma}\cdot\mathbf{e}\pi/2}=-e^{i\chi}\,\boldsymbol{\sigma}\cdot\mathbf{e},\quad U_{\mathrm{II}}=I \tag{4.3}$$

and obtain

$$\alpha_0=0,\quad \boldsymbol{\alpha}=-\mathbf{e},\quad \beta_0=1,\quad \boldsymbol{\beta}=0,\quad \mu=\chi.$$

The final intensity and polarization then is given by

$$I_f=I_i/2,$$

and

$$\mathbf{P}_f=\cos\chi\,\mathbf{e}\times\mathbf{P}_i-\sin\chi\,\mathbf{e}. \tag{4.4}$$

This implies that, with no phase shifter, the final polarization vector points in a direction which is orthogonal to both the initial polarization vector and to the rotation vector. By a additional nuclear phase shift, the polarization vector can be rotated in a plane orthogonal to the initial polarization. Thus, this example demonstrates some relationships between polarization and phases. Clearly, these effects could be readily measured by the use of an interferometer with a polarized neutron beam.

In the examples mentioned above the polarization vector of the final beam is confined to a plane orthogonal to the polarization directions of the two interfering beams which form the final beam. Therefore this experiment is important for the discussion of the problem of measurements [16], because here the final state has properties which neither of the two interfering states has, a result which could not be explained if the final beam would simply be regarded as a mixture of these states.

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