Analysis of the muon live-time data
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Statistical methods and tasks

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Contents

1 Introduction

One of the student experiments at HEPHY is the measurement of the muon live-time. Details for the setup of this experiment can be found in separate documents. In the following the focus will be on the analysis of the obtained data and their statistical interpretation. Although all examples will be for the measurement of the muon live time, the methods described in the following can easily be expanded for other measurements.

Just a short reminder: we are measuring the time difference between the time a particle enters our experiment and the time a particle leaves the system. The particle leaving the system can be a different particle than the initial one (e.g., a muon entering the system, decaying within, and an electron leaving the system). If we neglect any background sources, the measured time differences t originate in the decay of muons at rest. The decay of muons at rest is a so called Poisson process (i.e, the muons decay independent of each other, the probability for a decay in the time interval $[t_0, t_0 + dt]$ is independent of the time). Thus, the probability to observe k decays in a time interval $[t_0, t_0 + t]$ is given by the Poisson distribution:

$$P(N_\mu(t_0) - N_\mu(t_0 + t) = k) = \frac{e^{-\lambda_t}\lambda_t^k}{k!} \tag{1}$$

where λ_t is the expected number of decays in the time interval $[t_0, t_0 + t]$. The probability to observe no decay in the interval $[t_0, t_0 + t]$ is given by

$$P(N_\mu(t_0) = N_\mu(t_0 + t)) = e^{-\lambda_t} \tag{2}$$

The expected number of decays is given by the so-called live time τ_μ of the muon:

$$\lambda_t = (t_0 + t - t_0)\frac{1}{\tau_\mu} \tag{3}$$

$$\lambda_t = t\frac{1}{\tau_\mu} \tag{4}$$

Thus, the time until a muon decays follows the distribution

$$p(t) = e^{-\frac{t}{\tau_\mu}} \tag{5}$$

Note: The distribution of the time until a muon decays is independent of the start time. Thus, we can measure the muon live-time of cosmogenic muons although they are produced in the atmosphere some time before they enter our experiment.

2 Fitting data

To determine the muon live-time we are going to fit the theoretical distribution (5) to the measured distribution of time differences. This is achieved by maximising the likelihood $L(data|N_\mu, \tau)$ that a distribution like the measured one occurs, given that the muon live-time is τ and the number of observed events N_μ.

The measured time differences are discrete values (clocks cycles of a 10 MHz clock, see documentation of the experiment). Thus, in the following we will focus on binned fit methods. It is assumed, that the width of the time bins is $0.1\,\mu$s to match the 10 MHz clock. Later we will discuss the treatment of different bin widths.

2.1 Maximum likelihood fit

The histogram of the time differences has d bins. Each bin contains b_i entries. Since all bins are independent of each other, the likelihood to observe a histogram as the measured one given a muon live-time τ and N_μ events is given by

$$L(data|N_\mu,\tau) = \prod_{i=1}^{d} p(b_i|N_\mu,\tau) \tag{6}$$

where $p(b_i|N_\mu,\tau)$ are the probabilities that bin i contains b_i entries given a muon live-time τ and N_μ events.

In principle, the bin entries b_i of a histogram follow a Poisson distribution[1]. The likelihood is now given by

$$L(b_i|N_\mu,\tau) = \prod_{i=1}^{d} \frac{\lambda_i(N_\mu,\tau)^{b_i}}{b_i!} e^{-\lambda_i(N_\mu,\tau)} \tag{7}$$

$$\lambda_i(N_\mu,\tau) = N_\mu \int_{t_{i,low}}^{t_{i,up}} dt \frac{1}{\tau} e^{-\frac{t}{\tau}} \tag{8}$$

$$= N_\mu \left[exp\left(-\frac{t_{i,low}}{\tau}\right) - exp\left(-\frac{t_{i,up}}{\tau}\right)\right] \tag{9}$$

where λ_i is the expected number of entries for bin i, $t_{i,low}$ and $t_{i,up}$ are the lower and upper boundaries of bin i.

If another model is used for the fit, the integrand for the calculation of the λ_i has to be changed accordingly. It is also possible to use this formulation for multi-dimensional histograms, since the bins can still be numbered by one integer. Of course, the integral for the calculation of the expectations λ_i has to be changed to a multi-dimensional integral.

Now we have to find the maximum likelihood with respect to the free parameters N_μ and τ. For our problem (and most other cases) this can only be done numerically. Maximizing the logarithm of the likelihood is much more stable than maximizing the likelihood itself:

$$\ln(L(b_i|N_\mu,\tau)) = \sum_{i=1}^{d} b_i \ln(\lambda_i(N_\mu,\tau)) - \lambda_i(N_\mu,\tau) - \ln(b_i!) \tag{10}$$

There exists a variety of algorithms to minimize[2] functions in several dimensions we can use to find the best-fit values for the muon live-time τ and the number of events N_μ.

[1] This behaviour can change if the histogram is scaled or non-integer values are used for the bin-entries.

[2] Maximizing $\ln(L)$ is equivalent to minimizing $-\ln(L)$.

2.2 χ^2 - fit

The maximum likelihood fit discussed in the previous section can be applied for all fits to binned data in multiple dimensions, if the binned data represents counts. However, the calculation of the likelihood takes some computation time due to the logarithms and the factorial of the Poisson distributions of all bins. This problem can be avoided, if all bins have large entry numbers. For large expected numbers the Poisson distribution can be approximated by a normal distribution. The likelihood can then be written as

$$\begin{aligned} L(data|N_\mu,\tau) &= \frac{1}{\sigma_i\sqrt{2\pi}}e^{-\frac{(b_i-\mu_i(N_\mu,\tau))^2}{2\sigma_i^2}} && (11) \\ \mu_i(N_\mu,\tau) &= N_\mu \int_{t_{i,low}}^{t_{i,up}} \frac{1}{\tau}e^{-\frac{t}{\tau}} && (12) \\ &= N_\mu \left[exp\left(-\frac{t_{i,low}}{\tau}\right) - exp\left(-\frac{t_{i,up}}{\tau}\right)\right] && (13) \end{aligned}$$

where $\mu_i(N_\mu,\tau)$ is the mean value and $\sigma_i = \sqrt{\mu_i(N_\mu,\tau)}$ is the standard deviation of the normal distribution.

Similar to the maximum likelihood fit, this method can also be expanded to other models and multiple dimensions, by changing the integral for the calculation of the means μ_i.

For better numerical stability we have to maximize the logarithm of the likelihood instead of the likelihood itself:

$$\ln(L(data|N_\mu,\tau)) = \sum_{i=1}^{d} - \ln(\sigma_i\sqrt{2\pi}) - \frac{(b_i-\mu_i(N_\mu,\tau))^2}{2\sigma_i^2} \tag{14}$$

In the sum we still have to calculate a logarithm for each bin. However, we can perform an additional approximation: For large numbers we the standard deviation of the normal distribution $\sigma_i = \sqrt{\mu_i(N_\mu,\tau)}$ can be approximated by

$$\sigma_i \approx \sqrt{b_i} \tag{15}$$

This is justified since the relative width of the Poisson distribution ($\propto \frac{1}{\sqrt{N}}$) is small for large numbers. The square root of the mean value is close to the square root of the bin entries. With this approximation the logarithm of the likelihood is given as[3]

$$\ln(L(data|N_\mu,\tau)) = -\frac{d}{2}\ln 2\pi - \frac{1}{2}\sum_{i=1}^{d}\ln b_i - \frac{1}{2}\sum_{i=1}^{d}\frac{(b_i-\mu_i(N_\mu,\tau))^2}{\sigma_i^2} \tag{16}$$

Only the last part depends on the free parameters N_μ and τ. Thus, the logarithm of the likelihood is maximal with respect to N_μ and τ, if the last part is minimal. So instead of maximizing the logarithm of the likelihood we have to minimize the so-called χ^2:

$$\chi^2 = \sum_{i=1}^{d}\frac{1}{b_i}\left(b_i - \mu_i(N_\mu,\tau)\right)^2 \tag{17}$$

A χ^2 fit runs faster as a maximum-likelihood fit. However, it delivers wrong results if the bin entries are too small ($\lesssim 20$).

[3]All bins with zero entries are skipped ($\sigma_i = 0$).

3 Influence of different binnings

As already mentioned, the measured time differences are discrete clock cycles of a 10 MHz clock. If we use the same bin width (0.1 μs) this will not cause any problems. Figure 1 shows a time spectrum of simulated data with a bin width of

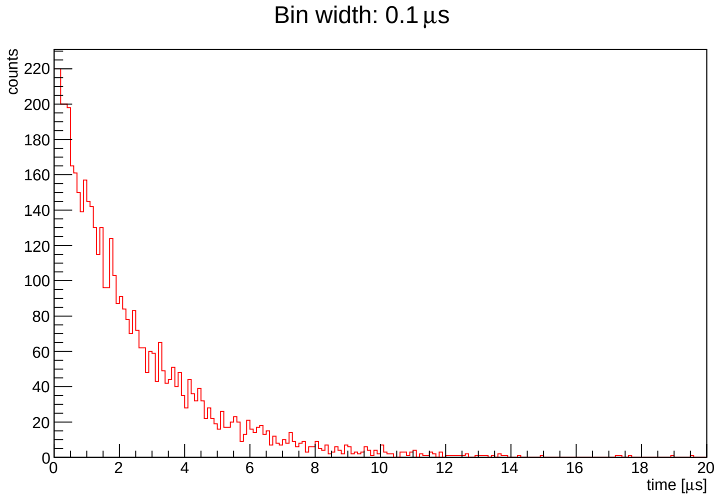

Figure 1: A simulated time-spectrum using a bin width of 0.1 μs

0.1 μs. Here the bin entries are indeed Poisson distributed. Thus, the methods explained in the previous section can be applied to infer the decay time.

However, if a different bin width is used, it can happen, that some bins contain significantly more or less events than other bins. This is depicted in

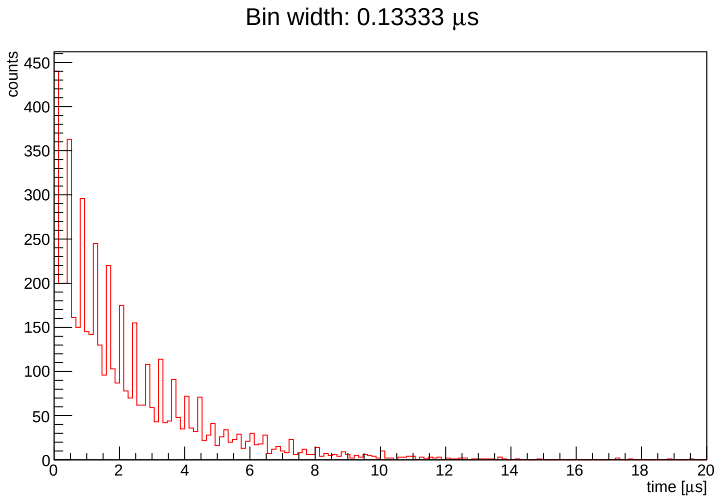

Figure 2: A simulated time spectrum using a bin width 0f 0.1333 μs

figure 2 where the bin width is increased to 0.1333 μs. In this case the bin entries are no longer Poisson distributed. Thus, the methods discussed in the previous section can not be applied.

It is possible to take this into account by including the discrete nature of the recorded times into the integrals for the expected bin contents. However, this procedure is rather complicated and error-prone.

The method discussed in the following will use random numbers to convert the acquired discrete times to continuous times. This method is easier to use and can be applied to a wider rage of problems without changing the method. Figure 3 visualizes the time-measurement procedure. First the muon is stopped

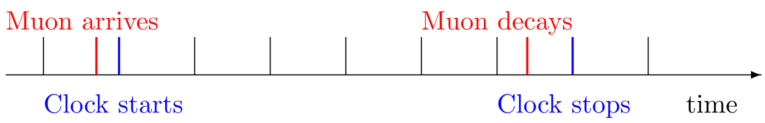

Figure 3: Visualization of the time measurement: The muon arrives and is stoped in the detector (red). The clock starts at the next available clock cycle (blue). The muon decays (red). The clock stops at the next clock cycle (blue).

in the detector (red). At the next clock cycle the time measurement (counter) is started. The measurement (counter) is stopped at the clock cycle (blue) succeeding the decay of the muon (red). Thus, the real time $t_{measured}$ between the muon entering the system and the decay of the muon is given by:

$$t_{real} = t_{measured} + \epsilon_{start} - \epsilon_{stop} \tag{18}$$

where $t_{measured}$ is the measured time (number of clock cycles), ϵ_{start} is the time difference between the muon entering the system and the start of the counter, and ϵ_{stop} is the time difference between the decay of the muon and the stop of the counter. We do not know the two time differences ϵ_{start} and ϵ_{stop}. However, we know that both differences are in the interval $[0, 0.1\,\mu\text{s}[$. Thus, to estimate the continuous time t_{real} we are choosing the time differences ϵ_{start} and ϵ_{stop} uniformly from the interval $[0, 0.1\,\mu\text{s}[$. We can use t_{real} to obtain the time-spectrum for the inference of the muon live time.

Figure 4 shows a comparison between the time-spectra obtained from the measured time values $t_{measured}$ (red) and the randomized times t_{real} (blue). For the blue spectrum the bin contents are again Poisson distributed and the fit methods discussed in the previous spectrum can be applied.

Of course an additional uncertainty is introduced by this method of estimating t_{real}. Its influence can be studied by repeating the randomization procedure and the fit several times and comparing the inferred values for the muon live time.

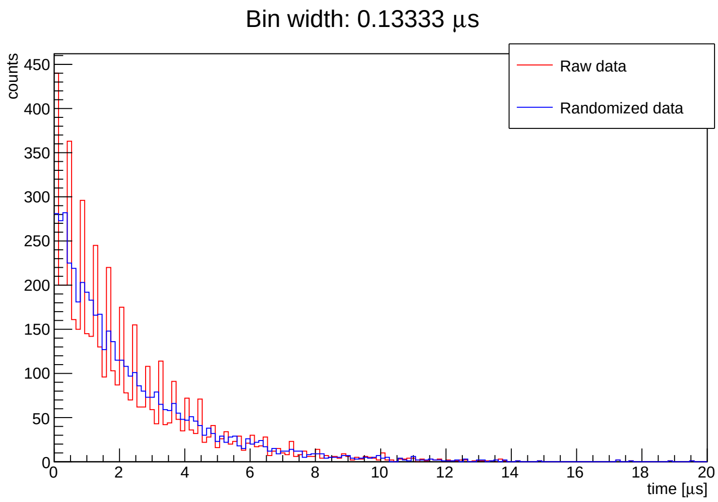

Figure 4: Comparison of the measured time-spectrum and a spectrum of the estimated (random) times t_{real}.

4 Tasks and questions

The following tasks and questions should be answered to obtain the measured muon live-time:

1. Simulate the time spectrum with the literature value for the muon live-time.
2. Which fit method should be used?
3. Apply the fit method to simulated data.
4. Can you reproduce the input value for the live time?
5. Try to estimate uncertainties by fitting a large number of simulated spectra
6. How to deal with backgrounds?
   - Determine the fit range
   - Model for the backgrounds?
7. Apply the updated fit method to measured data
8. How does the result compare to the literature value?
9. Does the result change for different binnings?
10. Apply the radomize method to the data to account for the binning effects
11. What are the statistical errors?
12. What are systematical errors?
13. Try to estimate the error due to the randomize method