WS 2017/18 JOSEF PRADLER

VO Field Theory Methods in Cosmology & Astroparticle Physics

I From Einstein's Eqs to Cosmology

II Standard Cosmology & shortcomings

III Inhomogeneous Universe & origin of structure

IV quantum fields at finite temperature

V* selected topics

Methods will be used in examples rather than being developed formally. Examples:

tensor algebra
principle of general covariance
variational principle
naive dimensional arguments
gauge invariance
perturbation theory in GR
second quantization (in gravitational background)
Bogoliubov transformations
statistical description of random fields*
quantum & thermal fluctuations
dispersion relations at finite T
resonances, narrow width approximation*

*time permitting.

I : From Einstein's Eqs to Cosmology

1. brief recap of tensor analysis (e.g. covariant deriv.)
2. Heuristic "derivation" of Einstein eqs.
3. Action principle in GR: Einstein Hilbert action
   Energy momentum tensor (+ proof of cons.)

II : Standard Cosmology & shortcomings

1. homogeneous & isotropic Universe (FRW spacetime)
2. cosmological solutions from Einstein eqs (Friedmann eqs.)
3. brief thermal history
4. flatness & horizon problem
5. inflation

III : The inhomogeneous Universe & origin of structure

1. perturbed Einstein eqs, classification of perturbations
2. gauge invariance in GR
3. scalar metric & matter perturbations
4.* statistical description of random fields in Cosmology
5. solution of EOM of perturbations (sketch); structure formation
6. Inflationary perturbations
7. more on observations

Plan for IV & V will follow.

Literature (to be amended)

I – III   Weinberg "Cosmology" & "Gravitation"
Landau & Lifshitz, Vol 2 Fields
Standard Cosmology is found in many textbooks
e.g. Kolb & Turner early Universe

1. Recap of tensor analysis / Elements of GR

There are 2 methods to find the effects of gravitation on physical systems:

1. write eqs. in locally inertial frame, then perform a coord. trafo, to find the eqs. in the frame of interest.
   (based on the principle of equivalence: laws of physics the same in any local Lorentz frame of curved spacetime as in any global Lorentz frame in flat spacetime.)
2. (elegant & convenient) Principle of general covariance: physical eqn. holds in a general grav. field, if
   a) holds in absence of gravity, i.e. for $g_{\mu\nu} \to \eta_{\mu\nu}$
   b) eqn. preserves form under general coord. trafo $x \to x'$

$\Rightarrow$ we need to know, how quantities transform under coord. trafo

<u>Scalars</u>: do not change under general coord. trafo. $x^\mu \to x'^\mu(x^\nu)$

$s'(x') = s(x)$ e.g. boost $x' = \Lambda x$ $\quad s'(\Lambda x) = s(x)$ or $s'(x) = s(\Lambda^{-1} x)$

<u>Vectors</u>:

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu \;; \quad V'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} V_\nu \;; \quad dx'^\mu = dx^\nu \frac{\partial x'^\mu}{\partial x^\nu}$$

"contravariant vector" "covariant vector"

Einstein summation convention always implied $A_\mu A^\mu = A_0 A^0 + A_1 A^1 + \ldots$

e.g. Euclidian rotation $V'^i(x') = R^{ij} V^j(x)$, or $V'^i(x) = R^{ij} V^j(R^{-1}x)$

<u>Tensors</u>: $T'^\mu{}_\nu = T^\rho{}_\lambda \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x^\lambda}{\partial x'^\nu}$ etc.

The most important example of a tensor is <u>metric tensor</u>

$$g_{\mu\nu} \equiv \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \qquad ; \; g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$$

ξ^α is a free falling coord. system, i.e. $\frac{d^2\xi^\alpha}{d\tau^2} = 0$,
with $d\tau^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta$ [a straight line in space]

Indices are raised & lowered with $g^{\mu\nu}$ & $g_{\mu\nu}$, resp.

$\Rightarrow$ large class of invariant eqs: LHS & RHS tensors with the same number of upper & lower tensor indices

Tensor density: transforms as a tensor, up to factors of Jacobian determinant

examples: $g = \det g_{\mu\nu}$

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} g_{\rho\sigma} \frac{\partial x^\sigma}{\partial x'^\nu} \quad \Rightarrow \text{take det} \qquad [\det(AB) = \det(A)\det(B)]$$

$$g' = \left|\frac{\partial x}{\partial x'}\right|^2 g \qquad \text{Jacobian}$$

important in the trafo of volume element

$$d^4x' = \left\|\frac{\partial x'}{\partial x}\right\| d^4x \quad \Rightarrow \quad \sqrt{-g}\, d^4x \text{ is invariant vol. el.}$$

Affine connection:

consider freely falling particle, only gravitational forces; principle of equivalence states that there is a coord. system ξ^α and that EOM implies straight line

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \qquad \text{with} \quad d\tau^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta$$

in an arbitrary coord. system $x^\mu(\xi^\alpha)$, this becomes

$$\frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\nu}{\partial \tau} = 0 \qquad \Big/ \cdot \frac{\partial x^\lambda}{\partial \xi^\alpha}$$

$$\Rightarrow \quad \frac{d^2x^\lambda}{d\tau^2} + \underbrace{\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}}_{\equiv \Gamma^\lambda_{\mu\nu}} \frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\nu}{\partial \tau} = 0 \quad \Big\} \text{ geodesic eqn.}$$

in general:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left[\frac{\partial g_{\rho\nu}}{\partial x^\mu} + \frac{\partial g_{\rho\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right]$$

i.e. $g_{\mu\nu}$ is the gravitational potential

$\Rightarrow$ moving backwards: equivalence principle states that there is a local Lorentz frame such that $g_{\mu\nu}(X) = \eta_{\mu\nu}$; $\partial_\gamma g_{\mu\nu}(X) = 0 \Rightarrow \Gamma = 0$

Covariant derivative:

$$\nabla_\lambda V^\mu = V^\mu{}_{;\lambda} \equiv \frac{\partial V^\mu}{\partial x^\lambda} + \Gamma^\mu_{\lambda\rho} V^\rho$$

$$\nabla_\lambda V_\mu = V_{\mu;\lambda} \equiv \frac{\partial V_\mu}{\partial x^\lambda} - \Gamma^\rho_{\mu\lambda} V_\rho$$

notably: $\nabla_\lambda g_{\mu\nu} = 0$ $\Rightarrow$ covariant diff. & raising/lowering indices commute

$$(g^{\mu\nu}V_\nu)_{;\lambda} = g^{\mu\nu}V_{\nu;\lambda}$$

cov. divergence: $\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g}\, V^\mu$

$\Rightarrow \int d^4x \sqrt{-g}\, \nabla_\mu V^\mu = 0$ covariant form of Gauss thm.

- definition of cov. deriv. such that tensors $\rightarrow$ tensors
- in the absence of gravitation $\Gamma \rightarrow 0$, $\nabla \rightarrow \partial$

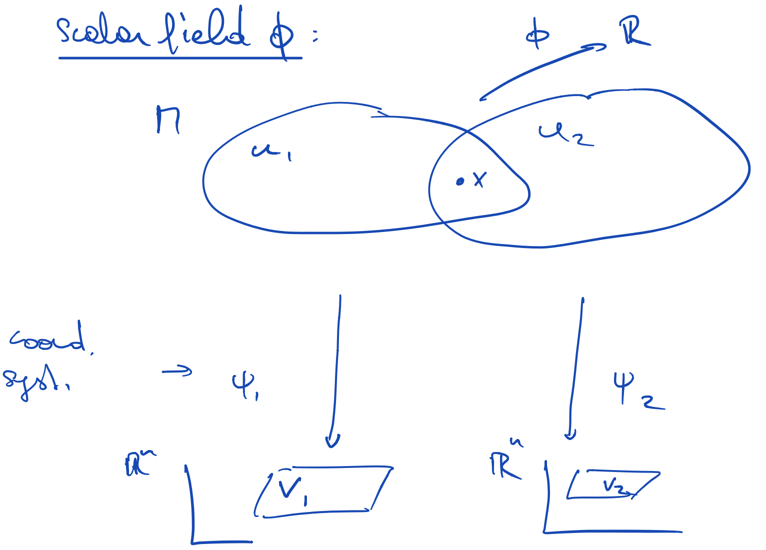
$\Rightarrow$ recipe for general covariance from special relat. eqs:

1) replace $\eta_{\mu\nu}$ by $g_{\mu\nu}$

2) replace ordinary derivatives with covariant deriv.

Addendum on coord. transfos (from the mathematical perspective)

Scalar field ϕ:



invariance under coord. transf $x \to x' = \Lambda x$

$$\phi'(x') = \phi(x)$$

equivalent to writing

$$\phi'(x) = \phi(\Lambda^{-1} x)$$

i.e. the transformed field at the new coord x' gives the same value as the original field at the point x

$\varphi_1 : U_1 \to \mathbb{R}^n$ $\qquad$ $\varphi_2 : U_2 \to \mathbb{R}^n$

Scalar field $\phi : M \to \mathbb{R}$

in coord representation φ_1

$$\phi_{\varphi_1} = \phi \circ \varphi_1^{-1} : V_1 \to \mathbb{R}$$

in coord. representation φ_2

$$\phi_{\varphi_2} = \phi \circ \varphi_2^{-1} : V_2 \to \mathbb{R}$$

$x \in U_1 \cap U_2$

$$\phi(x) = \phi \circ \varphi_1^{-1} \circ \varphi_1(x) = \phi_1(x_1)$$

$$\phi(x) = \phi \circ \varphi_2^{-1} \circ \varphi_2(x) = \phi_2(x_2)$$

such that $\phi_1(x_1) = \phi_2(x_2)$ $\Leftarrow$ SCALAR FIELD

Value of coord. representation ϕ_1 evaluated at the coord. representation $x_1 = \varphi_1(x)$ of the point x agrees with the value of the coord. representation $x_2 = \varphi_2(x)$ of the same point x. Scalar field is invariant under change of coordinates.

e.g. $\varphi_1(x) = x$; $\varphi_2(x) = \Lambda x + a$;

$$\phi_1(x) = \phi_2(\Lambda x + a)$$

or, with a change of notation : $\phi'(\Lambda x + a) = \phi(x)$, i.e. $\phi'(x') = \phi(x)$

2. "Derivation" of gravitational field eqs.

- Maxwell's eqs. linear, because EM-field (photon) does not carry charge
- Gravitational fields carry energy & momentum $\Rightarrow$ must contribute to their own source $\Rightarrow$ nonlinear PDE's
- derivation guided by Equivalence principle

local inertial frame at point X : $\begin{cases} g_{\mu\nu}(X) = \eta_{\mu\nu} \\ \partial_\gamma g_{\mu\nu}(x)\big|_{x=X} = 0 \end{cases}$ $\qquad \eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$ Minkowski

[$\partial_\alpha \partial_\beta g_{\mu\nu}(X) = 0$ not possible in curved space

$\Rightarrow$ in the vicinity of X we hope to describe the grav. field by <u>linear</u> PDE's, "weak field eqs." ;

$\Rightarrow$ we start in the Newtonian limit

EOM (Newton) $\quad \dfrac{d^2\vec{x}}{dt} = -\vec{\nabla}\phi \qquad \phi \dots$ Newtonian potential, for a point (or spherical) mass M

$$\phi = -GM/r \qquad r = |\vec{x}|$$

we compare this with EOM in GR

EOM (GR) $$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\ \ \rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$ (τ is the "proper time") $\quad d\tau^2 = g_{\mu\nu}\,dx^\mu dx^\nu$

↳ solve for $x(\tau)$ $\longrightarrow x(t) \longleftarrow$ solve for $t(\tau)$

a) slow particle $\Rightarrow$ neglect $d\vec{x}/d\tau$ compared to $dt/d\tau$

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\ \ 00}\left(\frac{dt}{d\tau}\right)^2 = 0$$

for ~~stationary~~ static grav. field, all time derivatives of $g_{\mu\nu}$ vanish:

$$\Gamma^\mu_{00} = -\frac{1}{2} g^{\mu\nu} \frac{\partial g_{00}}{\partial x^\nu}$$

o) now we make perturbative linear ansatz

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x) \qquad |h_{\mu\nu}| \ll 1$$

$$\Rightarrow \Gamma^\mu_{00} \simeq -\frac{1}{2} \eta^{\mu\nu} \frac{\partial h_{00}}{\partial x^\nu} \qquad \begin{cases} \Gamma^0_{00} \simeq -\frac{1}{2} \partial_\tau \underset{\substack{\text{stationary}\\ \text{field}}}{h_{00}} = 0 \\ \Gamma^i_{00} \simeq +\frac{1}{2} \partial_i h_{00} \end{cases}$$

$\Rightarrow$ linearized EOM

$$\underline{\mu = i} \quad \frac{d^2 \vec{x}}{d\tau^2} + \frac{1}{2} \vec{\nabla} h_{00} = 0 \qquad ; \qquad \underline{\mu = 0} \quad \frac{d^2 t}{d\tau^2} = 0$$

$$\Rightarrow \frac{d^2 \vec{x}}{dt^2} = -\frac{1}{2} \vec{\nabla} h_{00} \qquad \Rightarrow h_{00} = 2\phi + \text{const}$$

for $h_{00} \to 0$ for $|\vec{x}| \to \infty$

$\Rightarrow$ const $= 0$ for $\phi \to 0$

$$\Rightarrow \underline{g_{00} \doteq 1 + 2\phi} \qquad \text{(remember this eqn for later)}$$

o) Now we need to relate the metric perturbation to the matter content

$\Rightarrow$ Poisson eq.: $\nabla^2 \phi = 4\pi G \rho$ $\qquad$ ρ ... mass density

$$T_{00} \simeq \rho$$

$$\Rightarrow \underline{\nabla^2 g_{00} = 8\pi G T_{00}} \qquad \text{(to hold for weak static fields; not even L.-invariant)}$$

<u>GUESS</u>: Tensor-Eq: $\underline{G_{\mu\nu} = 8\pi G T_{\mu\nu}}$ $\longrightarrow$ for a general distribution $T_{\mu\nu}$ of energy & momentum

$G_{\mu\nu}$: linear comb. of metric & its derivatives

$$\left[\text{for point particles} \quad T^{\mu\nu} = \sum_i \frac{p_i^\mu p_i^\nu}{E_i} \delta^{(3)}(\vec{x} - \vec{x}_i(t)) \right]$$

What do we know further:

- $[T_{\mu\nu}] = \frac{\text{Energy}}{\text{Volume}}$, $[G] = \frac{1}{\text{Energy}^2}$ (NDA !)

  $\Rightarrow [G_{\mu\nu}] \sim \frac{1}{\text{Energy Volume}} \sim \frac{1}{\text{Area}} \Rightarrow$ dimension of a second derivative.

  $N =$ # derivatives $\Rightarrow N=2$

  terms with $N \neq 2$ must be multiplied by a constant of dimension $(\text{length})^{N-2}$

  $\Rightarrow$ $N > 2$ $(N<2)$ terms become negligible on sufficiently large (small) space-time scales.

  $\Rightarrow$ to get gravitational field eqs. that are uniform in scale: $N=2$

  $\Rightarrow$ $G_{\mu\nu}$ linear in second derivatives of $g_{\mu\nu}$ & quadratic in first derivatives

- $G_{\mu\nu}$ is symmetric, because $T_{\mu\nu}$ is

- covariant energy momentum conservation $\nabla_\mu T^\mu{}_\nu = 0$

  $\nabla_\mu G^\mu{}_\nu = 0$

- weak stationary field from non-relat. matter $G_{00} \simeq \nabla^2 g_{00}$

$\hookrightarrow$ enough to determine LHS:

Curvature tensor $R^\lambda{}_{\mu\nu\kappa}$ is the only tensor that can be constructed from $g_{\mu\nu}$ & its first & second derivatives & linear in second derivs.

$$R^\lambda{}_{\mu\nu\kappa} \equiv \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\kappa} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\nu} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta}$$

Ricci tensor: $R_{\mu\nu} \equiv R^\lambda{}_{\mu\lambda\nu}$

curvature scalar $R \equiv R^\mu{}_\mu$

} these are the only two tensors that can be formed.

$\Rightarrow G_{\mu\nu} = C_1 R_{\mu\nu} + C_2 g_{\mu\nu} R$

EX: • Bianchi Identities (= differential Id. on $R^\rho{}_{\mu\sigma\nu}$) fixes $C_2 = -\frac{C_1}{2}$

• weak field limit $C_1 = 1$

$$\Rightarrow G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$G_{\mu\nu}$ ... Einstein Tensor

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}}$$

Einstein Field Eqs.

Vacuum: $T_{\mu\nu} = 0$

contract E.Eq. with $g^{\mu\nu}$ $\Rightarrow R - 2R = +8G T^\mu{}_\mu \Rightarrow R = -8\pi G T^\mu{}_\mu$

use E.Eq. again: $R_{\mu\nu} + 4\pi G g_{\mu\nu} T^\alpha{}_\alpha = +8\pi T_{\mu\nu}$

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\alpha{}_\alpha\right)$$

$\Rightarrow$ in vacuum $R_{\mu\nu} = 0$ and gravitational fields exist in empty space

(in contrast to Minkowski for which $R^\rho{}_{\mu\nu\sigma} = 0$]

if we relax N=2 assumption, and allow for N<2 (not relevant on small scales)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Cosmological Constant (cc)

observation: $\Lambda \approx 10^{-120} m_p^2$ (the biggest mystery of modern physics)

we may put Λ on the RHS, and include it in $T_{\mu\nu}$ with

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \approx (10^{-3}\, eV)^4 \sim (\text{neutrino mass scale})^4$$

3. Action Principle in GR

- great advantage: immediate connection between symmetry principles & conservation laws

  in GR: symmetry is general covariance; conserv. is $T_{\mu\nu}$

- non-rigorous reminder on least action principle

  consider Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$ and action $S = \int d^4x\, \mathcal{L}$;
  change induced by $\phi(x) \to \phi(x) + \delta\phi(x)$ with $\delta\phi \to 0$ for $|x| \to \infty$
  in S is given by the difference

$$\delta S = \int d^4x\, \mathcal{L}(\phi + \delta\phi, \partial_\mu\phi + \delta(\partial_\mu\phi)) - \int d^4x\, \mathcal{L}(\phi, \partial_\mu\phi)$$

  $\Rightarrow$ expanded in powers of $\delta\phi$ & $\delta(\partial_\mu\phi)$; condition that S is extremal $\Leftrightarrow$ $\delta S = 0$

  note that $\delta(\partial_\mu\phi) = \partial_\mu \delta\phi$ commutes because we took δ at same x

  write $\phi'(x) = \phi(x) + \delta\phi(x)$, i.e. $\delta\phi(x) = \phi'(x) - \phi(x)$

$$\partial_\mu \delta\phi(x) = \partial_\mu(\phi'(x) - \phi(x)) = \partial_\mu\phi'(x) - \partial_\mu\phi(x) = \delta(\partial_\mu\phi(x)).$$

Variation $\phi \to \phi + \delta\phi$ leads to

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) \right]$$

$$= \int d^4x \left\{ \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right]\delta\phi + \underbrace{\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right]\}}_{\text{total derivative, i.e. value depends on boundary cond.; we only consider } \phi \to 0 \text{ for } |x| \to \infty}$$

$$\frac{\delta S}{\delta \phi} = 0 \iff \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$$ Euler-Lagrange Eqs.

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e.g. $S = \int d^4x \left[|\partial_\mu \phi|^2 - m^2 \phi \phi^* \right] \Rightarrow (\Box + m^2)\phi = 0$

$(\Box + m^2)\phi^* = 0$

Klein-Gordon eq.

If $\mathcal{L}$ is invariant under special type of variation $\phi \to \phi + \alpha \delta\phi$ with some small parameter α, up to total divergence $\mathcal{L}(x) \to \mathcal{L}(x) + \alpha \partial_\mu K^\mu$

$$\underset{\substack{\uparrow \\ \text{by assumption}}}{\delta S = 0} = \alpha \int d^4x \left\{ \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] + \partial_\mu \underbrace{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right]}_{\text{identify with } K^\mu} \right\}$$

$$\Rightarrow j^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - K^\mu \quad \Rightarrow \partial_\mu j^\mu = 0 \quad ; \quad Q \equiv \int j^0 d^3x = \text{const}$$

Noether Current — local consv. — global consv.

e.g. for the complex scalar above $\mathcal{L}$ is invariant under

$$\phi \to e^{-i\alpha}\phi \qquad \alpha \in \mathbb{R}$$

or, infinitesimally $\phi \to \phi - i\alpha\phi$, $\phi^* \to \phi^* + i\alpha\phi^*$

$$\Rightarrow \delta\phi = -i\phi \quad ; \quad K^\mu = 0$$

$$j^\mu = -i(\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi)$$

$$\partial_\mu j^\mu = -i(\phi \Box \phi^* - \phi^* \Box \phi) = 0$$ by eom.

$\Box\phi = -m^2\phi$

$\Box\phi^* = -m^2\phi^*$

Examples:

- geodesic eqn. from action principle

  consider freely falling particle from $A \to B$

$$S_{AB} = \int_A^B ds = \int_A^B \frac{ds}{d\sigma}\, d\sigma = \int \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}}\, d\sigma$$

proper time $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$

parameter describing path $x^\mu(\sigma)$

now, consider variation of $x^\mu(\sigma) + \delta x^\mu(\sigma)$ with $\delta x^\mu|_{A,B} = 0$

EX: $\delta S_{AB} = 0$. Show that the geodesic eqn. follows.

- Energy-Momentum tensor from variation of the metric

  consider S_M the action of a material system, and take infinitesimal variation

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$$

$\hookrightarrow$ arbitrary, except that it vanishes for $|x^\mu| \to \infty$

$$\delta S_M = -\frac{1}{2}\int d^4x \sqrt{-g(x)}\, T^{\mu\nu}(x)\, \delta g_{\mu\nu}$$

serves as definition of the energy momentum tensor

[N.B. sign of def. changes if $T_{\mu\nu}\delta g^{\mu\nu}$ is used because $\delta g^{\mu\nu} = -g^{\mu\varsigma}g^{\nu\sigma}\delta g_{\varsigma\sigma}$ (see later)]

Einstein's Eqs. from the action principle :

$$S = S_G + S_M$$

Einstein-Hilbert action ┘ └ Matter-part

$$S_G = -\frac{1}{16\pi G}\int d^4x \sqrt{-g(x)}\, R(x) \qquad g = \det g_{\mu\nu}$$

Now we perform variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ and demand, that the action is stationary $\delta S = 0$.

$$\delta S_G = -\frac{1}{16\pi G}\,\delta\int d^4x \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}$$

$$= -\frac{1}{16\pi G}\int d^4x \left\{ (\delta\sqrt{-g})\, g^{\mu\nu} R_{\mu\nu} + \sqrt{-g}\,(\delta g^{\mu\nu}) R_{\mu\nu} + \sqrt{-g}\, g^{\mu\nu} (\delta R_{\mu\nu}) \right\}$$

We need $\delta\sqrt{-g} = \delta\sqrt{-\det g_{\mu\nu}} = -\frac{1}{2}\frac{\delta g}{\sqrt{-g}}$

$$\delta \ln\det M = \ln\det(M+\delta M) - \ln\det M \qquad \text{(difference)}$$

$$= \ln\frac{\det(M+\delta M)}{\det M} = \ln\det\left(M^{-1}(M+\delta M)\right)$$

$$= \ln\det(1 + M^{-1}\delta M) \qquad \text{(near identity, det behaves like trace)}$$

use $\det e^A = e^{\mathrm{Tr}\,A}$

$$\approx \ln(1 + \mathrm{Tr}\, M^{-1}\delta M)$$

$$\approx \mathrm{Tr}\, M^{-1}\delta M \qquad (\ln(1+\varepsilon) \approx \varepsilon)$$

$$\Rightarrow \delta \ln g = \frac{\delta g}{g} = \mathrm{Tr}\, g^{\mu\nu}\delta g_{\nu\rho} = g^{\mu\nu}\delta g_{\nu\mu}$$

$$\Rightarrow \delta\sqrt{-g} = -\frac{1}{2} g\, g^{\mu\nu}\delta g_{\nu\mu}\frac{1}{\sqrt{-g}} = +\frac{1}{2}\sqrt{-g}\, g^{\rho\sigma}\delta g_{\rho\sigma} = -\frac{1}{2}\sqrt{-g}\, g_{\rho\sigma}\delta g^{\rho\sigma}$$

here we had the typo in the lecture

o) $\delta g^{\mu\nu}$: use $\delta(g^{\mu\rho}g_{\rho\nu}) = 0 = \delta g^{\mu\rho}g_{\rho\nu} + g^{\mu\rho}\delta g_{\rho\nu}$

$$g_{\rho\nu}\,\delta g^{\mu\rho} = -g^{\mu\rho}\,\delta g_{\rho\nu} \qquad /\cdot g^{\nu\sigma}$$
$$\delta^{\sigma}_{\rho}\,\delta g^{\mu\rho} = -g^{\nu\sigma}g^{\mu\rho}\,\delta g_{\rho\nu}$$
$$\underline{\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\,\delta g_{\rho\sigma}}$$

o) $\delta R_{\mu\nu}$: let's go into local inertial frame, i.e. $\Gamma^{\mu}_{\nu\rho}=0$, find the variation there & use principle of general covariance
(justified because although Γ is not a tensor, $\delta\Gamma$ is as it measures the difference of two vectors that have been parallel-transported to the same point, one by Γ, the other by $\Gamma+\delta\Gamma$; difference of a vector is a vector)

$$R_{\mu\nu} = \frac{\partial\Gamma^{\rho}_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial\Gamma^{\rho}_{\mu\rho}}{\partial x^{\nu}} + \underbrace{\Gamma^{\rho}_{\mu\nu}\Gamma^{\alpha}_{\rho\alpha} - \Gamma^{\alpha}_{\mu\rho}\Gamma^{\rho}_{\nu\alpha}}_{=0 \text{ local inertial frame}}$$

$$\Rightarrow \delta R_{\mu\nu} = \frac{\partial\delta\Gamma^{\rho}_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial\delta\Gamma^{\rho}_{\mu\rho}}{\partial x^{\nu}}$$

$\Rightarrow$ covariant version $\quad \delta R_{\mu\nu} = \nabla_{\rho}\,\delta\Gamma^{\rho}_{\mu\nu} - \nabla_{\nu}\,\delta\Gamma^{\rho}_{\mu\rho}$ $\quad$ Palatini Identity

$$\Rightarrow \sqrt{g}\,g^{\mu\nu}\,\delta R_{\mu\nu} = \sqrt{-g}\,g^{\mu\nu}\left[\nabla_{\rho}\,\delta\Gamma^{\rho}_{\mu\nu} - \nabla_{\nu}\,\delta\Gamma^{\rho}_{\mu\rho}\right]$$
$$= \sqrt{-g}\left[\nabla_{\rho}\left(g^{\mu\nu}\,\delta\Gamma^{\rho}_{\mu\nu}\right) - \nabla_{\nu}\left(g^{\mu\nu}\,\delta\Gamma^{\rho}_{\mu\rho}\right)\right]$$

then, using $\nabla_{\mu}V^{\mu} = \frac{1}{\sqrt{-g}}\partial_{\mu}\sqrt{-g}\,V^{\mu}$

$$= \partial_{\rho}\left(\sqrt{-g}\,g^{\mu\nu}\,\delta\Gamma^{\rho}_{\mu\nu}\right) - \partial_{\nu}\left(\sqrt{-g}\,g^{\mu\nu}\,\delta\Gamma^{\rho}_{\mu\rho}\right)$$
$= $ total derivative

Return to variation:

$$S = S_G + S_M \quad ; \quad \delta S = 0$$

$$\Rightarrow \delta S_G = -\frac{1}{16\pi G}\int d^4x \left\{ (\delta\sqrt{-g})\, g^{\mu\nu} R_{\mu\nu} + \sqrt{-g}(\delta g^{\mu\nu}) R_{\mu\nu} + \sqrt{-g}\, g^{\mu\nu} (\delta R_{\mu\nu}) \right\}$$

$$= -\frac{1}{16\pi G}\int d^4x \left\{ \frac{1}{2}\sqrt{-g}\, g^{\rho\sigma} R - \sqrt{-g}\, g^{\mu\rho} g^{\nu\sigma} R_{\mu\nu} \right\} \delta g_{\rho\sigma} + \text{tot. deriv.}$$

$$= -\delta S_M = +\frac{1}{2}\int d^4x \sqrt{-g}\, T^{\rho\sigma}(x)\, \delta g_{\rho\sigma}$$

$$\Rightarrow \quad R^{\rho\sigma} - \frac{1}{2} g^{\rho\sigma} R = 8\pi G T^{\rho\sigma}$$ Einstein Field Eqs

Energy-Momentum tensor - definition & proof of conservation

- The matter action $S_m = \int d^4x \sqrt{-g}\, \mathcal{L}(\phi, \partial_\mu \phi)$ is a scalar, hence should not change under coord. trafos: $\delta S_m = 0$ for $x^\mu \to x'^\mu$

- Consider coord. trafo $x'^\mu = x^\mu + \xi^\mu(x)$ with small ξ

$\Rightarrow$ changes are induced in ϕ as $\delta\phi$ and in $g_{\mu\nu}$ as $\delta g_{\mu\nu}$

$$\delta S_m = \delta_\phi S_m + \delta_g S_m$$

$\delta_\phi S_m = 0$ by the virtue of the EOM, which are obtained from this condition

$\Rightarrow$ we need to work out $\delta_g S_m$

$$\delta_g S_m = \delta_g \int d^4x \sqrt{-g}\, \mathcal{L}\left(\phi, \partial_\mu\phi, g_{\mu\nu}, \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right)$$

$$= \int d^4x \left(\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial \frac{\partial g^{\mu\nu}}{\partial x^\rho}} \delta \frac{\partial g^{\mu\nu}}{\partial x^\rho} \right)$$

$$\overset{\text{P.I.}}{=} \int d^4x \underbrace{\left(\frac{\partial \sqrt{-g}\,\mathcal{L}}{\partial g^{\mu\nu}} - \frac{\partial}{\partial x^\rho} \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial \frac{\partial g^{\mu\nu}}{\partial x^\rho}} \right)}_{\equiv +\frac{1}{2}\sqrt{-g}\, T_{\mu\nu}} \delta g^{\mu\nu}$$

(see note on the sign above)

$$\Rightarrow \delta S_m = +\frac{1}{2} \int d^4x \sqrt{g}\, T_{\mu\nu}\, \delta g^{\mu\nu} = 0 \qquad \text{if } S_m \text{ is a scalar}$$

Requiring $\delta S_m = 0$ does not imply $T_{\mu\nu} = 0$ because 10 $\delta g^{\mu\nu}$ components are not independent, but result from $x'^\mu = x^\mu + \xi^\mu$, i.e. only 4 comp.

Proof, that $T_{\mu\nu}$ is conserved when general covariance holds, i.e. when $\delta S_m = 0$:

1) obtain explicit form of $\delta g^{\mu\nu}$ in coord. trafo $x \to x + \xi$;

2) use it in $\delta S_m = 0$

The metric tensor under coord trafo changes as

$$g'^{\mu\nu}(x') = g^{\rho\sigma}(x) \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}}$$

$$= g^{\rho\sigma}(x) \left(\delta^{\mu}_{\rho} + \frac{\partial}{\partial x^{\rho}} \xi^{\mu} \right) \left(\delta^{\nu}_{\sigma} + \frac{\partial}{\partial x^{\sigma}} \xi^{\nu} \right)$$

$$\simeq g^{\mu\nu}(x) + g^{\mu\sigma} \frac{\partial \xi^{\nu}}{\partial x^{\sigma}} + g^{\nu\rho} \frac{\partial \xi^{\mu}}{\partial x^{\rho}}$$

We would like to express $g'^{\mu\nu}(x')$ in terms of the old coordinates

$$g'^{\mu\nu}(x') = g'^{\mu\nu}(x + \xi) \simeq g'^{\mu\nu}(x) + \xi^{\lambda} \frac{\partial}{\partial x^{\lambda}} g'^{\mu\nu}(x)$$

$$\simeq g'^{\mu\nu}(x) + \xi^{\lambda} \frac{\partial}{\partial x^{\lambda}} g^{\mu\nu}(x) + \text{higher order in } \xi$$

$$\Rightarrow g'^{\mu\nu}(x) = g^{\mu\nu}(x) \underbrace{- \xi^{\lambda} \frac{\partial}{\partial x^{\lambda}} g^{\mu\nu}(x) + g^{\mu\sigma} \frac{\partial \xi^{\nu}}{\partial x^{\sigma}} + g^{\nu\rho} \frac{\partial \xi^{\mu}}{\partial x^{\rho}}}_{\nabla^{\mu} \xi^{\nu} + \nabla^{\nu} \xi^{\mu}}$$

EX:

$\Rightarrow$ transformation law of $g^{\mu\nu}$ under infinit. coord. trafo $x^{\mu} \to x^{\mu} + \xi^{\mu}(x)$

$$g'^{\mu\nu}(x) = g^{\mu\nu}(x) + \nabla^{\mu} \xi^{\nu}(x) + \nabla^{\nu} \xi^{\mu}(x) = g^{\mu\nu} + \delta g^{\mu\nu}$$

where $\delta g^{\mu\nu} = \nabla^{\mu} \xi^{\nu} + \nabla^{\nu} \xi^{\mu}$

N.B.: $\nabla^{\mu} \xi^{\nu} + \nabla^{\nu} \xi^{\mu} = 0$ are called "Killing eqs." $\Rightarrow$ coord. trafos ξ that leave metric invariant.

now, we can use this in the variation of S_m :

$$\delta S_m = \frac{1}{2}\int d^4x \sqrt{-g}\, T_{\mu\nu}\, \delta g^{\mu\nu}$$

$$= \frac{1}{2}\int d^4x \sqrt{-g}\, T_{\mu\nu}\left(\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu\right)$$

$$= \int d^4x \sqrt{-g}\, T_{\mu\nu}\, \nabla^\mu \xi^\nu \qquad \text{, since } T_{\mu\nu} \text{ is symmetric}$$

$$= \int d^4x \sqrt{-g}\left[\nabla_\mu\left(T^{\mu\nu}\xi_\nu\right) - \xi_\nu \nabla_\mu T^{\mu\nu}\right] \qquad \text{(indices pulled)}$$

recall cov. divergence of a vector $\nabla_\mu V^\mu = \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{-g}\,V^\mu)}{\partial x^\mu}$

$$= \underbrace{\int d^4x \frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\, T^{\mu\nu}\xi_\nu\right)}_{\text{total derivative}} - \int d^4x\, \xi_\nu \nabla_\mu T^{\mu\nu}$$

$$\Rightarrow \delta S_m = -\int d^4x\, \xi_\nu \nabla_\mu T^{\mu\nu} \overset{!}{=} 0$$

Since ξ_ν is arbitrary

$$\nabla_\mu T^{\mu\nu} = 0 \qquad \text{covariantly conserved.}$$

$$T_{\mu\nu} = +\frac{2}{\sqrt{-g}}\left[\frac{\partial \sqrt{-g}\,\mathcal{L}}{\partial g^{\mu\nu}} - \frac{\partial}{\partial x^\rho}\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial \frac{\partial g^{\mu\nu}}{\partial x^\rho}}\right]$$

in absence of a grav. field $\nabla_\mu T^{\mu\nu} = 0 \Rightarrow \frac{\partial}{\partial x^\mu} T^{\mu\nu} = 0$

Analogy with EM

The proof that general covariance leads to energy-momentum conservation has exact analogon in EM : gauge invariance leads to conservation of EM current

$$\text{consider } S = S_M \underbrace{- \frac{1}{4}\int d^4x \sqrt{-g}\, F_{\mu\nu}F^{\mu\nu}}_{= S_F} \qquad \left[F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\right]$$

under trafo $A_\mu \to A_\mu + \delta A_\mu$ $\quad \delta S_F = 0$, but S_M may transform

again, we identify $\delta S_M = \int d^4x \sqrt{-g}\, \underbrace{J^\mu(x)}\, \delta A_\mu(x)$

definition of EM current

As before we shall consider a particular trafo that leaves S_M invariant; called a gauge transformation, under which $\delta S_M = 0$

$$A_\mu \to A_\mu + \partial_\mu \alpha(x)$$

$$\Rightarrow \delta A_\mu = \partial_\mu \alpha(x)$$

inhomogeneous transformation; (analogous with Christoffel-Symbols that did not transform as a tensor)

matter fields of charge e:

$$\psi(x) \to e^{ie\alpha(x)}\psi(x)$$

like in gravity where we found that derivative of a tensor does not yield a tensor, derivative $\partial_\mu \psi(x)$ does not transform like $\psi(x)$

$$\partial_\mu \psi(x) \to e^{ie\alpha(x)}\left(\partial_\mu \psi(x) + ie\,\psi(x)\,\partial_\mu \alpha(x)\right)$$

Cure again is to introduce covariant derivative

$$D_\mu \psi = \left[\partial_\mu - ieA_\mu\right]\psi(x)$$

$$\Rightarrow D_\mu \psi \to (D_\mu \psi)\, e^{ie\alpha(x)}$$

$$\delta S_m = \int d^4x \sqrt{-g}\, J^\mu(x)\, \frac{\partial \alpha(x)}{\partial x^\mu} \overset{!}{=} 0$$

if action is to be called gauge-invariant.

$$\overset{P.I.}{=} -\int d^4x\, \alpha(x)\, \frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\, J^\mu\right) \overset{!}{=} 0$$

since α is arbitrary, $\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\, J^\mu\right) = \nabla_\mu J^\mu = 0$

gauge invariance implies covariant conservation of EM-current.