

## IV. Particles in media

(Largely follows Roffet "Grosses Laboratoires for New Physics")

### I. Motivation:

in astrophysics, physical processes often proceed inside a dense or hot medium (inside star, plasma etc...) ; propagation of particles (photons, neutrinos, ...) effected ; even completely new excitations may become possible  
⇒ leads to a study of dispersion relations of particles inside media ; For example,  $\gamma \rightarrow \nu\bar{\nu}$  becomes possible

In field theory, we often describe particles by a plane wave expansion

$e^{-i(\omega t - \vec{k} \cdot \vec{r})}$  with frequency  $\omega$  & wave vector  $\vec{k}$

$\omega$  is determined by the dispersion relation

In vacuum, Lorentz invariance implies  $\omega^2 - \vec{k}^2 = m^2$ , since  $p = (\omega, \vec{k})$  is a four-vector; e.g. phase space for  $\gamma \rightarrow \nu\bar{\nu}$  is  $\propto \omega^2 - \vec{k}^2$

In media, dispersion relations are generally modified by the coherent interactions with the background.

as photons in non-relativistic plasmas acquire a medium-induced effective mass  $\omega^2 = \omega_p^2 + \vec{k}^2$

$\omega_p^2 = 4\pi \hbar n_e / m_e$  is the plasma frequency ;  
 $n_e$  is the electron density.

⇒ for  $\omega_p > 2m_\gamma$   $\gamma \rightarrow \nu\bar{\nu}$  is kinematically possible

- if neutrinos carry e.g. magnetic dipole moment, they can contribute to the cooling of stars  $\Rightarrow$  derive limits on NDM's etc.
- Higgs mechanism can be viewed as "refractive" phenomena with universal dispersion relation  $E^2 = m^2 + \vec{p}^2$
- "space-like" dispersion relations with  $p = (E, \vec{p})$  and  $p^2 = E^2 - \vec{p}^2 < 0$  are possible, amounting to negative effective "tachyonic" mass squared  $\tilde{p}^2 = M_{\text{eff}}^2 < 0$

In isotropic media, dispersion relation often expressed as

$$|\vec{k}| = n_{\text{refr}} \omega \quad n_{\text{refr}} \dots \text{refractive index}$$

$n_{\text{refr}} > 1$  ... corresponds to space-like excitations;  
e.g.  $e \rightarrow e\gamma$  kinematically allowed  
(Cherenkov radiation)

- in addition to modified dispersion relations, new excitations can occur, e.g. longitudinal polarizations of EM-field "plasmons";  $\gamma_L \rightarrow \nu\bar{\nu}$  also possible

There is a whole formalism "linear field theory" for treating these effects; we, however, will choose simpler approach

[e.g. lectures at TU Wien by A. Rebhan]

dispersion relations in QFT at  $T=0$  e.g. discussed at length in Bjorken & Drell

## 2. Particle dispersion : refractive index

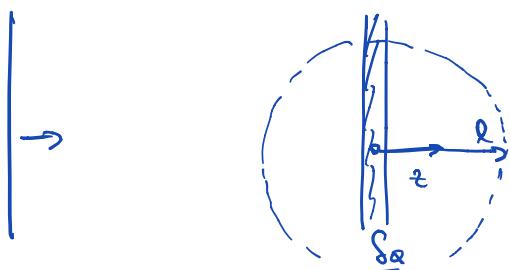
dispersion relations can be obtained from forward scattering amplitude [see e.g. Sоловьев "Advanced QM"]  
in QR

$$\phi(\vec{r}, t) \propto e^{-i\omega t} \left( e^{i\vec{k} \cdot \vec{r}} + f(\omega, \theta) \frac{e^{i\omega r}}{r} \right) \quad r = |\vec{r}|$$

$\uparrow$   
 scattering amplitude  
 $k = |\vec{k}|$

differential cross section :  $\frac{d\sigma}{d\Omega} = |f(\omega, \theta)|^2$

consider plane wave in  $z$  direction, traveling relativistically in vacuum,  $\omega = k$ , incident onto a slab of infinitesimal thickness  $S\alpha$  with  $N$  scattering centers / unit volume



cylindrical coord.

$$g = \sqrt{x^2 + y^2}; \theta = \arctan \frac{y}{x}$$

transmitted wave at  $z = l$  (relativistic with neglecting time dependence  $e^{-i\omega t}$ )

$$\phi(l) \propto e^{i\omega l} + N S\alpha \int_0^\infty \frac{e^{i\omega \sqrt{g^2 + l^2}}}{\sqrt{g^2 + l^2}} f(\omega, \theta) 2\pi g dg$$

Integration by parts yields at a distance  $l \gg \frac{1}{\omega}$

$$\phi(l) \propto e^{i\omega l} \left[ 1 + \frac{2\pi i}{\omega} N S\alpha f(\omega) \right] + O\left(\frac{1}{\omega}\right) \quad (*)$$

with forward scattering amplitude  $f(\omega) = f(\omega, \theta=0)$

(\*) for making the integral convergent  $\omega \rightarrow \omega + i\epsilon$  with  $\epsilon > 0$  in the exponent  
since the integrand oscillates even for large  $\rho$

for a slab of finite thickness  $a$  obtained by summing infinitesimal areas  
with  $\delta x = a/j$  and 'slip limit'  $j \rightarrow \infty$

$$\lim_{j \rightarrow \infty} \left[ 1 + i \frac{2\pi N a}{j \omega} f(\omega) \right]^j = e^{i(2\pi a/\omega) f a}$$

putting this back into the previous expression

$$\phi(l) \propto e^{i\omega l} e^{i(2\pi N/\omega) f a} \\ e^{i\omega l \left[ 1 + \frac{2\pi N}{\omega^2} \frac{a}{l} f(\omega) \right]}$$

$\Rightarrow$  over a distance  $a$ , a phase shift  $e^{i N_{\text{refl}} \omega a}$  accumulates, where

$$n_{\text{refl}} = 1 + \frac{2\pi}{\omega^2} N f(\omega) \quad \text{is the index of reflection}$$


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Examples:

o) Consider Thomson scattering  $g_e \rightarrow g_e$  non-relat.

$$\frac{d\sigma}{d\Omega} = \frac{e^2}{m_e^2} |\boldsymbol{\Sigma} \cdot \boldsymbol{\Sigma}'|^2 \quad \boldsymbol{\Sigma} \dots \text{scattering vector}$$

Forward scattering:  $|\boldsymbol{\Sigma} \cdot \boldsymbol{\Sigma}'|^2 = 1 \Rightarrow f(\omega) = + \underbrace{\frac{e^2}{m_e}}_{\substack{\text{sign needs} \\ \text{actual} \\ \text{calculation}}}$

$\Rightarrow$  Dispersive relation  $k^2 = n_{\text{eff}}^2 \omega^2 \approx \omega^2 + 4\pi N f(\omega) + O(L^2)$

$\Rightarrow \omega^2 = k^2 + \frac{4\pi N e}{m_e} \leftarrow \text{electron density}$

$$\overbrace{= \omega_p^2}^{\text{Plasma frequency of a non-relat.}} \frac{\text{plasma } (T \ll m_e)}{}$$

↳ In general  $n_{\text{eff}}$  is complex

consider the attenuation of an EM-wave

$$\text{Intensity} \propto e^{-l/\lambda_{\text{mfp}}} = e^{-\sigma_{\text{tot}} \cdot N \cdot l} \quad \lambda_{\text{mfp}} = \frac{l}{\sigma_{\text{tot}} \cdot N}$$

Plane wave amplitude (again drop  $e^{i\omega t}$  factor)

$$e^{ikl - l/2\lambda_{\text{mfp}}} = e^{ikl - \sigma_{\text{tot}} N l/2} = e^{i n_{\text{eff}} \omega l}$$

$$\Rightarrow k = (\text{Re } n_{\text{eff}}) \omega ; \quad (2\lambda_{\text{mfp}})^{-1} = (\text{Im } n_{\text{eff}}) \omega$$

$$\sigma_{\text{tot}} = 2 \text{Im } n_{\text{eff}} \omega = \frac{4\pi \text{Im } f(\omega)}{\omega} \quad \text{optical thin. (exact)}$$

⇒ Imaginary part of  $n_{\text{eff}}$  describes the absorption

For completeness, we mention that causality implies certain analytical properties of  $n_{\text{eff}}$  ( $f(\omega)$  has to be analytical in the upper complex  $\omega$ -plane), such that Re & Im part are related by

$$\text{Re } n_{\text{eff}}(\omega) = 1 + \frac{2}{\pi} \int_0^\infty \frac{\omega' \text{Im } n(\omega') d\omega'}{\omega'^2 - \omega^2}$$

Kramers-Kronig relation (O(ω)-relation) [see Sekhri ad. QN]

3. Wave fct. renormalizable:

- The primary effect of modified dispersion relations is the particle kinematics, i.e. the phase space; example of  $\gamma \rightarrow \nu\bar{\nu}$  was given
  - matrix elements can also be affected by what a "particle" is inside a medium

Consider a free scalar field  $\phi$ . Its spatial Fourier components  $\phi_{\vec{k}}$  satisfy the harmonic oscillator equation of motion

$$\ddot{\phi}_{\vec{n}} + (\vec{k}^2 + m^2) \underbrace{\phi_{\vec{n}}}_{\equiv \omega_{\vec{n}}^2} = 0$$

In the last lecture, we saw that canonical quantisation with  $\tilde{\omega} = \phi$  leads to quantised energies  $\hbar\omega_{\pm}$  ( $\omega_+$  is the classical frequency for the mode  $\tilde{\omega}$  as dictated by the dispersion relation)

⇒ for an excitation with given  $\omega \approx$  "field strength"  
 that determines the coupling strength to a source  
 (e.g. photon to electron) is modified

Consider now a particle inside a medium. Klein Gordon eqn then reads 4-velocities

$$\left[ -\kappa^2 + \overleftarrow{T} T_k(\omega) \right] \phi_k = g g_k(\omega)$$

↓  
self-energy      coupling to external source

$$k = (\omega, \vec{k})$$

$$\Pi_{\vec{u}}(\omega) = \Pi(\omega, \vec{u})$$

$\Pi = m^2 + \text{medium induced term calculated for } f(\omega)$

homogeneous egn with  $g=0$

$$\Rightarrow k^2 = \Pi_{\vec{z}}(\omega) \Rightarrow \text{dispersion relation } \omega_{\vec{z}}^2 - \vec{k}^2 = \Pi_{\vec{z}}(\omega_{\vec{z}})$$

implicit egn that determines the frequency  $\omega_{\vec{z}}$  of a freely prop. wave with wavevector  $\vec{k}$

Since  $\Pi_{\vec{z}}(\omega)$  can be a complicated fn, this implies "dispersion", i.e. in wave space the egn. of  $\phi(x)$  is not the simple second order differential egn.  $\ddot{\phi}_{\vec{z}} + (\vec{k}^2 + \Pi_{\vec{z}}) \phi_{\vec{z}} = g g_{\vec{z}}$  as this would lead to usual particles with energies( $t$ )  $\omega_{\vec{z}}$

Taylor-Expand around  $\omega_{\vec{z}}$ :

$$\Pi_{\vec{z}}(\omega) = \Pi_{\vec{z}}(\omega_{\vec{z}}) + \Pi'_{\vec{z}}(\omega_{\vec{z}})(\omega - \omega_{\vec{z}}) \quad \Pi' = \partial \omega \Pi$$

$\Rightarrow$  Klein Gordon egn becomes

$$[-\omega^2 + \vec{k}^2 + \underbrace{\Pi_{\vec{z}}(\omega_{\vec{z}}) + \Pi'_{\vec{z}}(\omega_{\vec{z}})(\omega - \omega_{\vec{z}})}_{=\omega_{\vec{z}}^2 - \vec{k}^2}] \phi_{\vec{z}} = g g_{\vec{z}}(\omega)$$

$$\left. [-\omega^2 + \omega_{\vec{z}}^2 + \Pi'_{\vec{z}}(\omega_{\vec{z}})(\omega - \omega_{\vec{z}})] \right\} \phi_{\vec{z}}(\omega) = g g_{\vec{z}}(\omega)$$

Write as:

$$\frac{1}{2} (\omega_{\vec{z}}^2 - \omega^2) \phi_{\vec{z}}(\omega) = g g_{\vec{z}}(\omega)$$

with  $\frac{1}{2} = \frac{-\omega^2 + \omega_{\vec{z}}^2 + \Pi'_{\vec{z}}(\omega - \omega_{\vec{z}})}{\omega_{\vec{z}}^2 - \omega^2}$

$$\begin{aligned}
 &= \frac{-(\omega - \omega_{\vec{z}})(\omega + \omega_{\vec{z}}) + \text{Tr}_{\vec{z}'}(\omega - \omega_{\vec{z}})}{-(\omega - \omega_{\vec{z}})(\omega + \omega_{\vec{z}})} \\
 &\approx \frac{2\omega_{\vec{z}} - \text{Tr}_{\vec{z}'}}{2\omega_{\vec{z}}} = 1 - \frac{\partial \text{Tr}}{\partial \omega^2} \Big|_{\omega^2 - k^2 = \text{Tr}(\omega, \vec{z})}
 \end{aligned}$$

Given  $\vec{k}$ ,  $Z$  is a constant so that the Hamiltonian to one above is

$$H = H_0 + \text{Int} = \frac{1}{2} \frac{1}{Z} (\dot{\phi}_{\vec{z}}^2 + \omega_{\vec{z}}^2 \phi_{\vec{z}}^2) + g \phi_{\vec{z}} \rho_{\vec{z}}$$

rescale  $\phi$ , to get to standard harmonic osc.  $\phi_{\vec{z}} = \sqrt{Z} \tilde{\phi}_{\vec{z}}$

$$\begin{aligned}
 &= \frac{1}{2} (\dot{\tilde{\phi}}_{\vec{z}}^2 + \omega_{\vec{z}}^2 \tilde{\phi}_{\vec{z}}^2) + \sqrt{Z} g \tilde{\phi}_{\vec{z}} \rho_{\vec{z}}
 \end{aligned}$$

$\Rightarrow$  inside a medium, particle interacts with modified strength

$\Rightarrow$  in Feynman graphs, a factor  $\sqrt{Z}$  for every external  $\phi$ -field needs to be included

## 4. Photon dispersion in a plasma

Recall Maxwell's eqs in covariant form

$$\partial_\mu F^{\mu\nu} = j^\nu \text{ where } j^\nu = (g, \vec{j}), F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

$$\text{or, in terms of } A : D A_\nu - \partial_\nu (D_\mu A^\mu) = j_\nu$$

Gauge invariance: Maxwell's eqs invariant under  $A_\mu \rightarrow A_\mu - \partial_\mu d$

$$\text{we may write } A_\mu = (\phi, \vec{A}) \Rightarrow \vec{E} = -\vec{\nabla}\phi - \dot{\vec{A}}, \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\text{Two important gauges are } \begin{cases} \partial_\mu A^\mu = 0 & \text{Lorentz-gauge} \\ \vec{\nabla} \cdot \vec{A} = 0 & \text{Coulomb-gauge} \end{cases}$$

$$\Rightarrow \text{Maxwell's eqs : } \begin{cases} D\phi = g & D\vec{A} = \vec{j} \\ -\vec{\nabla}^2\phi = g & D\vec{A} = \vec{j}_T \quad \vec{\nabla} \cdot \vec{j}_T = 0 \end{cases} \begin{cases} \text{Lorentz} \\ \text{Coulomb} \end{cases}$$

$$\text{for } g=0, \vec{j}=0 : \phi=0 \text{ (Coulomb)}, D\vec{A}=0 \text{ in both} \\ \Rightarrow \underbrace{(\omega^2 - \vec{k}^2)}_{\Rightarrow \omega^2 = \vec{k}^2} \vec{A} = 0 \\ \Rightarrow \omega^2 = \vec{k}^2 \text{ Dispersion relation} \\ \text{of a massless particle}$$

Inside a medium, we include the medium-induced polarization tensor  $\Pi^{\mu\nu}$ , so that Maxwell's eqs in Fourier space become

$$(-k^2 g^{\mu\nu} + k^\mu k^\nu + \Pi^{\mu\nu}) A_\nu = j_\mu^{\text{ext}} \quad \Pi^{\mu\nu} = \Pi^{\mu\nu}(\omega, \vec{k})$$

$- j_\mu^{\text{ind}}$  induced current, "linear response"

consider gauge field  $A_\nu \rightarrow A_\nu + K_\nu \omega$ . Induces a charge  
 $\Rightarrow -K^2 K^\nu \omega + K^\nu K^2 \omega + \Pi^{\nu\mu} K_\mu \omega = 0$

$$\Rightarrow \Pi^{\nu\mu} K_\mu = 0$$

because of current conservation  $K_\mu J^\mu_{\text{ext}} = 0$ ,  $K_\mu (J^\mu_{\text{in}} + J^\mu_{\text{ext}}) = 0$   
 $\Rightarrow K_\mu \Pi^{\mu\nu} = 0$

In vacuum,  $\Pi^{\mu\nu}$  can only be constructed from  $g^{\mu\nu}$  &  $K^\mu K^\nu$  each of which violates  $\Pi^{\mu\nu} K_\mu = 0$ ; This forbids a mass-less  $m^2 g^{\mu\nu}$  to be part of  $\Pi^{\mu\nu}$

for  $J_{\text{ext}} = 0$ , the equation of  $\phi$  in Coulomb gauge becomes

$$(\vec{k}^2 + \Pi^{00}) \phi = 0$$

$\Rightarrow$  dispersion relation  $\vec{k}^2 + \Pi^{00}(\omega, \vec{k}) = 0$

$\Rightarrow$  E-field associated with  $\phi$  is  $\propto \vec{k} \phi$  along prop. direction,  
i.e. longitudinal excitation

(physically it is a density wave of electrons, like a sound-wave)

## 5. $\Pi^{\mu\nu}$ in an isotropic medium

In addition to  $g^{\mu\nu}$  &  $K^\mu u^\nu$  there is also four-velocity of the medium  $u^\mu$  to contract  $\Pi^{\mu\nu}$ ;  $u^\mu = (1, \vec{0})$  in the rest frame of the Riemann ball

$$\Pi^{\mu\nu} = -\Pi_T \sum_{i=T,L} \Sigma_{T,i}^{\mu} \Sigma_{T,i}^{\nu} - \Pi_L \Sigma_L^{\mu} \Sigma_L^{\nu}$$

↑  
transverse  
pol. vectors



Wording: various definitions of  
how the product is splitted

There are in fact 3  $\Pi_i$ 's in optically active media (R & L handed plane pols. experience different resp.). Here we only consider the case  $\Pi_T = \Pi_R = \Pi_L$

$\Pi^{\mu\nu}$  is gauge invariant, but def's of  $\Sigma_L$  &  $\Pi$ 's differ and also depend on gauge

$\Sigma_i^{\mu}, \Sigma_i^{\nu}$  are projectors onto the transverse and longitudinal subspaces  
We shall pick Lorentz gauge in the following  $k_\mu A^\mu = 0$

longitudinal pol.  $\Sigma_L = \frac{1}{\sqrt{\omega - \vec{k}^2}} \begin{pmatrix} |\vec{k}| \\ \omega \vec{k} \end{pmatrix} \hat{k} = \frac{\vec{k}}{|\vec{k}|}$

transverse pol. (e.g. circular)  $\Sigma_{\pm} = \begin{pmatrix} 0 \\ \hat{\Sigma}_{\pm} \end{pmatrix} \quad \hat{\Sigma}_{\pm} = (\hat{e}_x \pm i \hat{e}_y) / \sqrt{2}$

$$\sum_{i=T,L} \Sigma_i^{\mu} \Sigma_i^{\nu} = -g^{\mu\nu} + \frac{k_T k_\nu}{k^2}$$

⇒ can be used to project to L & T modes

$$\Pi_i = \Sigma_i^{\perp} \Pi_{\text{perp}} \Sigma_i^{\perp}$$

$$\Pi_L = -\frac{\omega^2 - \vec{k}^2}{\vec{k}^2} \Pi^{\parallel\parallel}$$

$$\Pi_T = \frac{1}{2} (\Pi \Pi - \Pi_L)$$

Maxwell eqns become

$$(k^2 \rho^{\perp\perp} - \sum_{i=L,T} \Pi_i \Sigma_i^{\perp} \Sigma_i^{\perp}) A_{\perp} = 0$$

⇒ dispersion relation for physical excitation  $A_{\perp} = A_0 \Sigma^{\perp}$

$$\omega^2 - \vec{k}^2 - \Pi_{\perp}(\omega, |\vec{k}|) = 0$$

This implicit equation yields frequency  $\omega_n$  for an excitation of a given pol. & wave number.

"effective mass"  $m_{\text{eff}}^2 = \Pi_{\perp}(\omega_n, |\vec{k}|)$  (omega real & negative)

Converted to macroscopic EM:

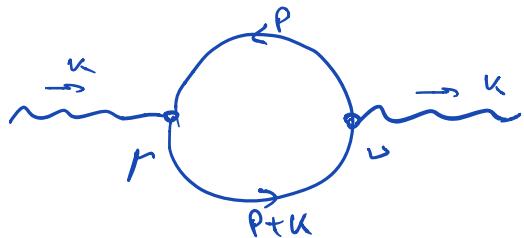
$$\vec{D} = \epsilon \vec{E} \quad \epsilon \text{--- dielectric permittivity}$$

$$\vec{H} = \mu^{-1} \vec{B} \quad \mu \text{--- magnetic permeability}$$

one may choose  $\vec{H} = \vec{B}$ ,  $\vec{D}_T = \epsilon_T \vec{E}_T$ ,  $\vec{D}_L = \epsilon_L \vec{E}_L$  ( $\vec{k} \cdot \vec{D}_T = 0$ )

$$\epsilon_L = 1 - \frac{\Pi_L}{\omega^2 - \vec{k}^2} \quad \epsilon_T = 1 - \frac{\Pi_T}{\omega^2} \quad \Rightarrow \quad \epsilon_L(\omega, |\vec{k}|) = 0$$

## 6. $\Pi^{\mu\nu}$ to lowest order in QED



$$\Pi^{\mu\nu}(K) = (-i)(-ie)^2 \int \frac{d^4 p}{(2\pi)^4} T_F [g^\mu g^\nu S_F(p) g^\rho g^\sigma S_F(p+K)]$$

In thermal field theory, the finite temperature fermion propagator is

$$S_F(p) = (p+m) \left[ \frac{1}{p^2 - m^2 + i\epsilon} + \frac{2\pi i \delta(p^2 - m^2)}{e^{(p+u)/T} + 1} \right]$$

Separating the usual vacuum polarization from the finite T part

$$\Pi^{\mu\nu}(K) = \Pi_{T=0}^{\mu\nu}(K) + \Pi_{T>0}^{\mu\nu}$$

The finite T-part in the next term of the Ward loop becomes  
(after rescaling S-fct and shifting  $p+K$ )

see. e.g. Ahmed, Nasrood Ann. Phys. 207 (1981)

$$\begin{aligned} \Pi_{T>0}^{\mu\nu} &= 16\pi d \int \frac{d^3 p}{2E_p (2\pi)^3} \left[ \frac{1}{e^{(E_p+K)/T} + 1} + \frac{1}{e^{(E_p-K)/T} + 1} \right] \\ &\times \frac{(p \cdot K)^2 g^{\mu\nu} + K^2 p^\mu p^\nu - (p \cdot K)(K^\mu p^\nu + K^\nu p^\mu)}{(p \cdot K)^2 - \frac{1}{4}(K^2)^2} \end{aligned}$$

$$E_p = \sqrt{\vec{p}^2 + m^2} \quad \mu \dots \text{chemical potential}$$

$$[\dots] = n_F(E_p) + \bar{n}_F(E_p) \quad \text{net phase space density of electrons + positrons}$$

The integration can be done analytically, when  $(k^2)^2$  term in the denominator is neglected

⇒ OK in nonrel. limit  $m_e \gg T$  ( $P.K \approx m_e \cdot \omega$ )

⇒ OK in rel. limit when near light cone  $\omega \approx |\vec{k}|$

By a lucky coincidence it turns out that neglecting  $(k^2)^2$  term is actually more accurate. The reason is that it removes the unphysical contribution  $\gamma \rightarrow e^+e^-$  which never becomes kinematically possible (else  $e^\pm$  receive thermal energy) [dropping  $(k^2)^2$  removes imaginary part of amplitude]

$$\Pi_L = \frac{4\lambda}{\pi} \frac{\omega^2 - \vec{k}^2}{\vec{k}^2} \int dP (n_F + \bar{n}_F) \frac{\vec{p}^2}{E_p} \left[ \frac{\omega}{|\vec{k}|v} \ln \frac{\omega + \vec{k}v}{\omega - \vec{k}v} \right] - \frac{\omega^2 - \vec{k}^2}{\omega^2 - \vec{k}^2 v^2} - 1$$

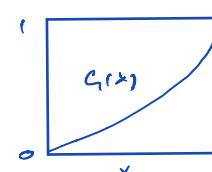
$$\Pi_L = \omega_p^2 \left[ 1 - G \left( \frac{v_*^2 k^2}{\omega^2} \right) \right] + \sigma_*^2 k^2 - k^2 \quad |\vec{k}| = k$$

$$\Pi_T = \omega_p^2 \left[ 1 + \frac{1}{2} G \left( \frac{v_*^2 k^2}{\omega^2} \right) \right] \quad \text{holds in } \underline{\text{all}} \text{ cases}$$

$$v_* = \begin{cases} \sqrt{5T/m_e} & \text{Classical} \\ v_F & \text{Degenerate} \\ 1 & \text{Relativistic} \end{cases} \quad \dots \text{typical electron velocity}$$

$$\omega_p^2 = \begin{cases} \frac{4\pi k n_e}{m_e} \left( 1 - \frac{S}{2} \frac{T}{m_e} \right) & \text{Classical} \\ \frac{4\pi k n_e}{E_F} = \frac{4\lambda}{3\pi} P_F^2 v_F & \text{Degenerate} \quad P_F \dots \text{fermi moment} \\ \frac{4\lambda}{3\pi} (p^2 + \frac{1}{3} \pi^2 T^2) & \text{Relativistic} \end{cases}$$

For completeness:

$$\left\{ \begin{array}{l} \omega_* = \frac{\omega_1}{\omega_p} \\ \omega_p^2 = \frac{4\pi}{\pi} \int_0^\infty d\rho (n_F + \bar{n}_F) \rho \left( \nu - \frac{1}{3}\nu^3 \right) \\ \omega_1^2 = \frac{4\pi}{\pi} \int_0^\infty d\rho (n_F + \bar{n}_F) \rho \left( \frac{5}{3}\nu^3 - \nu^5 \right) \\ G(x) = \frac{3}{x} \left[ 1 - \frac{2x}{3} - \frac{1-x}{2\sqrt{x}} \ln \left( \frac{1+\sqrt{x}}{1-\sqrt{x}} \right) \right] \\ G(0) = 0, G(1) = 0 \end{array} \right.$$


Dispersion relations from  $\omega^2 - k^2 = \pi_{L\pi}(\omega, k)$  (kunzederhol eqs.)

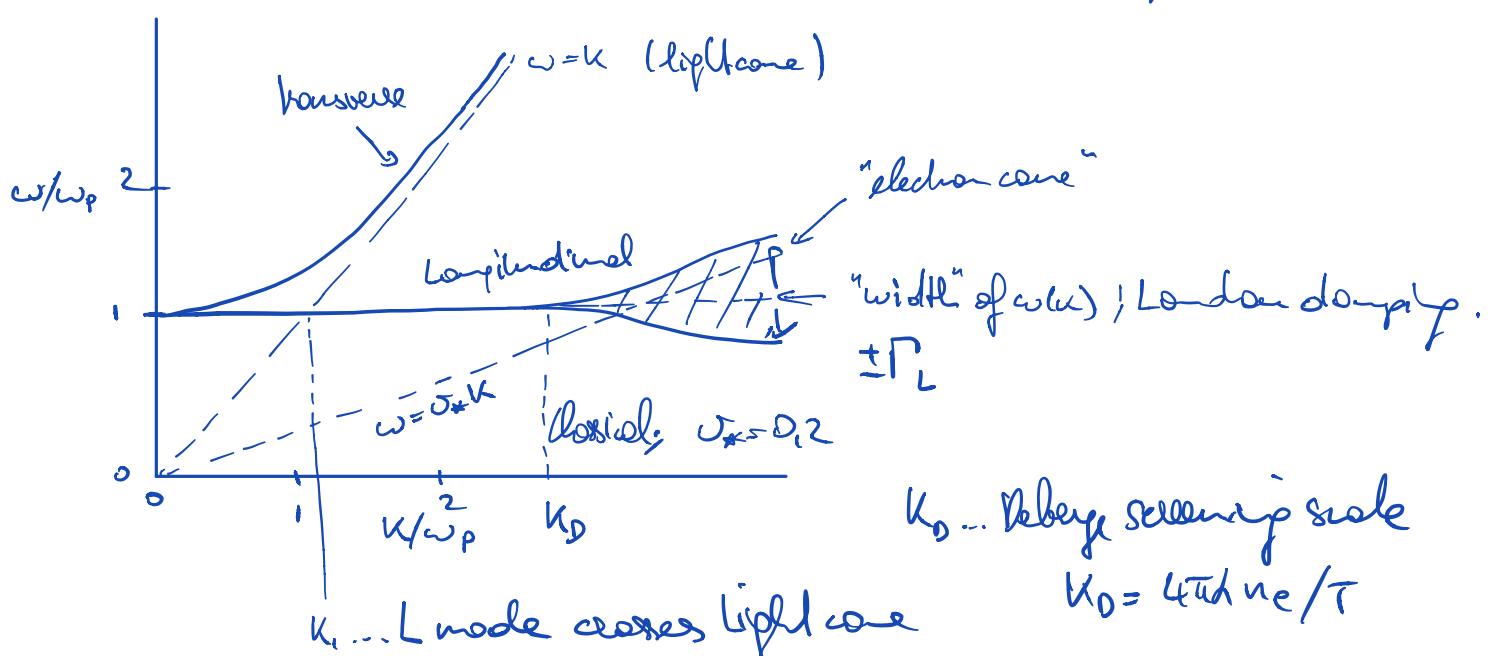
Classical limit in  $O(T/m_e)$

$$\omega^2 = k^2 + \omega_p^2 \left( 1 + \frac{k^2}{\omega^2} \frac{T}{m_e} \right) \quad \text{Transverse}$$

$$\omega^2 = \omega_p^2 \left( 1 + 3 \frac{k^2}{\omega^2} \frac{T}{m_e} \right) \quad \text{Longitudinal}$$

$T \ll m_e$ : T-modes like massive particles

L-modes oscillate with fixed frequency indep. of  $k$



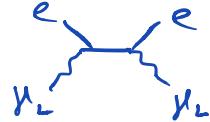
$$K_1 = \begin{cases} 1 + 3T\ln \epsilon & \text{Classical} \\ \infty & \text{Relativistic} \end{cases}$$

$\Rightarrow$  for  $K > K_1$ , the 4-momentum is space-like  $\omega^2 - \vec{K}^2 < 0$  for L-modes

T-modes stay light-like in  $\epsilon$  plasma (no Cherenkov process exise)  
 L-modes get Cherenkov absorbed;  $\text{Im } \Pi_L \neq 0$

In the expression for  $\Pi_{T\mu\nu}$  this corresponds to  $P \cdot K = 0$  for  $(K^2)^2 = 0$

$P \cdot K = E_p \omega - \vec{p} \cdot \vec{K}$  can vanish if  $K > \omega$ ;  $\Pi_L$  becomes imaginary;  
 one can think of it as the electron propagator in the  
 Compton scattering going on-shell



Recall:

Consider the part of a propagator  $\text{Im } z = \frac{1}{2i} (z - z^*)$

$$\text{Im } \frac{1}{p^2 - m^2 + i\Sigma} = \frac{1}{2i} \left( \frac{1}{p^2 - m^2 + i\Sigma} - \frac{1}{p^2 - m^2 - i\Sigma} \right) = \frac{-\Sigma}{(p^2 - m^2)^2 + \Sigma^2}$$

$\rightarrow 0$  for  $\Sigma \rightarrow 0$  except near  $p^2 = m^2$

if we integrate over  $p^2$

$$\int_0^\infty d p^2 \frac{-\Sigma}{(p^2 - m^2)^2 + \Sigma^2} = -\bar{\Gamma}$$

$$\Rightarrow \text{Im } \frac{1}{p^2 - m^2 + i\Sigma} = -\bar{\Gamma} \delta(p^2 - m^2)$$

propagator is real, except possible goes on-shell.]

## 7. Screening of interactions

Starting on a Coulomb potential (e.g. Rutherford sc., Poisson) has divergence in forward direction, because of the long-range nature of the electrostatic interaction.

In Plasma: divergence regulated by screening

Consider static ( $\omega = 0$ ) limit of Maxwell's eqs.

$$[\vec{k}^2 + \Pi_L(0, k)] \phi(k) = g(k)$$

$$[\vec{k}^2 + \Pi_T(0, k)] \vec{A}(k) = \vec{j}(k) \quad \vec{k} \cdot \vec{j} = 0$$

In vacuum, eqn. for  $\phi$  is F.T. of Poisson-eqn  $\Delta \phi = g$

$$\Rightarrow \text{for } g(k) = e S(k) \Rightarrow \phi(k) = \frac{1}{k}$$

$$\underline{\Pi_T(0, k) = 0}$$

stationary currents  $\partial_t \vec{j} > 0$  are not screened; Plasma does not affect value of magnetic field

$$\Pi_L(0, k) = \frac{4\pi}{\Omega} \int_0^\infty dp (n_f + \bar{n}_f) p \left( \nu + \frac{1}{\nu} \right) \equiv K_D^2 \quad \text{indep of } k$$

$$\Rightarrow (\vec{k}^2 + \vec{k}_D^2) \phi(k) = g(k)$$

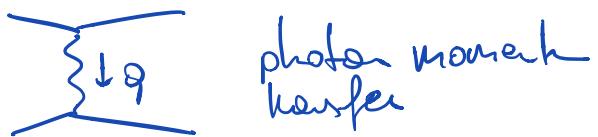
$\Rightarrow$  yields Yukawa potential for point-like source

$$\phi(k) = \frac{1}{k} e^{-\lambda k_D} \quad \begin{array}{l} \text{electric charges are screened} \\ \text{for } \lambda > K_D \end{array}$$

$$k_D^2 = \frac{4\pi d n_e}{T} = \frac{m_e}{T} \omega_p^2 \quad \text{indep of } m_e \\ \Rightarrow \text{all ions contribute}$$

In astrophysics this is often accounted for by replacing the propagator

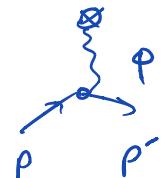
$$\frac{1}{|\vec{q}|^4} \rightarrow \frac{1}{(\vec{q}^2 + k_D^2)^2}$$



photon moment transfer

Ok procedure, when the scattering is "slow" such that charges exchange themselves; averages all first, squares after (a precise prescription is obtained by studying correlations in a plasma)

e.g. Rutherford scattering  $\frac{d\sigma}{d\Omega} = \frac{4(z_\alpha)^2 E^2}{|\vec{q}|^4}$



$$|\vec{q}|^2 = 2|\vec{p}|^2 (1 - \cos\theta)$$

energy loss of a particle when traversing a plasma

$$\int d\cos\theta \underbrace{(1 - \cos\theta)}_{\Delta E \text{ in a scattering}} \frac{d\sigma}{d\Omega} \xrightarrow{\text{summed}} \ln \Lambda$$

"Coulomb-log"

$$\ln \Lambda = \ln \left( \frac{4\bar{p}^2}{k_D^2} \right) - 1$$

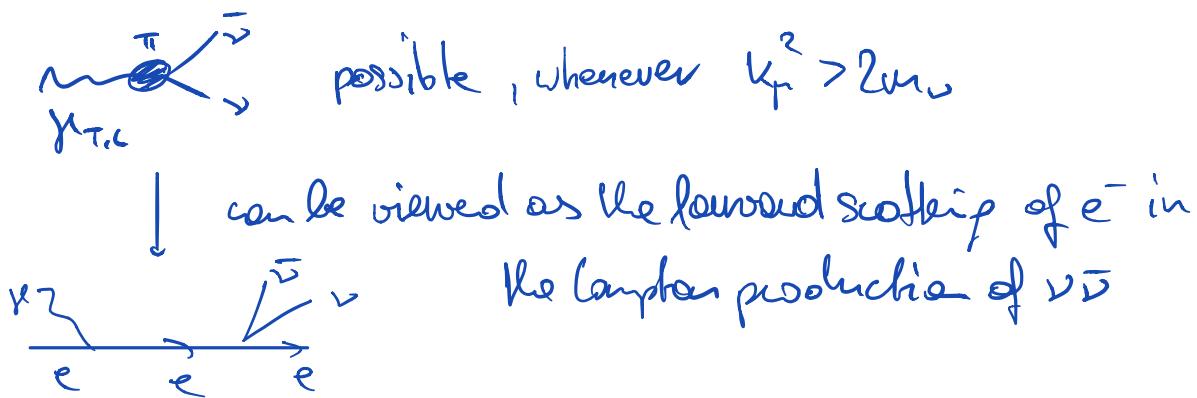
$$\text{or } \ln \left( \frac{r_0}{b_{\min}} \right)$$

$\rightarrow$  impact parameter

$$b_{\min} = \max \{ b_{\min}^{el}, b_{\min}^{ion} \}$$

Price-question: What happens in gravity?

## 8. "Plasma" decay into $\nu\bar{\nu}$ : (means here T&L)



$$\text{NC: } J_{\text{int}} = -\frac{G_F}{2} \bar{e} \gamma_\mu (c_V + c_A \gamma_5) e \bar{\nu} \gamma^\mu (1 - \gamma^5) \nu$$

$$\text{EM: } L_{\text{int}} = -e \bar{e} \gamma_\mu e A^\mu$$

response of the medium to EM-excitation is  $J_{\text{ind}} = \nabla A$ ; coherent electric oscillations serve as a source for  $\nu$ -current.

Only  $c_V$  contributes

$$\Rightarrow \text{effective current: } \frac{c_V G_F}{e\sqrt{2}} A^\mu(k) \Pi_{T,L}(k) \bar{\nu} \gamma^\mu (1 - \gamma^5) \nu$$

$\uparrow \quad \downarrow$   
one factor already contained in  $\mu_{\mu\nu}$

$$\text{decay rate: } \Gamma = \int \frac{d^3 p}{(2\pi)^3 2E_p} \frac{d^3 \bar{p}}{(2\pi)^3 2E_{\bar{p}}} (2\pi)^4 \delta^4(k - p - \bar{p}) \frac{1}{2\omega} \sum_{\text{spins}} |\mathcal{M}|^2$$

$\uparrow \quad \downarrow$   
 $\nu, \bar{\nu}$  momenta

$$\sum |\mathcal{M}|^2 = \mu_{\mu\nu} p^\mu \bar{p}^\nu$$

$$\mu_{\mu\nu} = \frac{4G_F^2 C_V^2}{e^2} Z_{T,L} \Pi_{T,L}^2 (\rho_{\mu\nu} + 2 \sum_{\gamma}^* \epsilon_{\mu\nu})$$

$$\Rightarrow \Gamma_{T,L} = \frac{C_V^2 G_F^2}{48\pi^2 \omega} Z_{T,L} \frac{(\omega^2 - \vec{k}^2)^3}{\omega} \quad \text{where } \omega^2 - \vec{k}^2 = \Pi_{T,L}$$

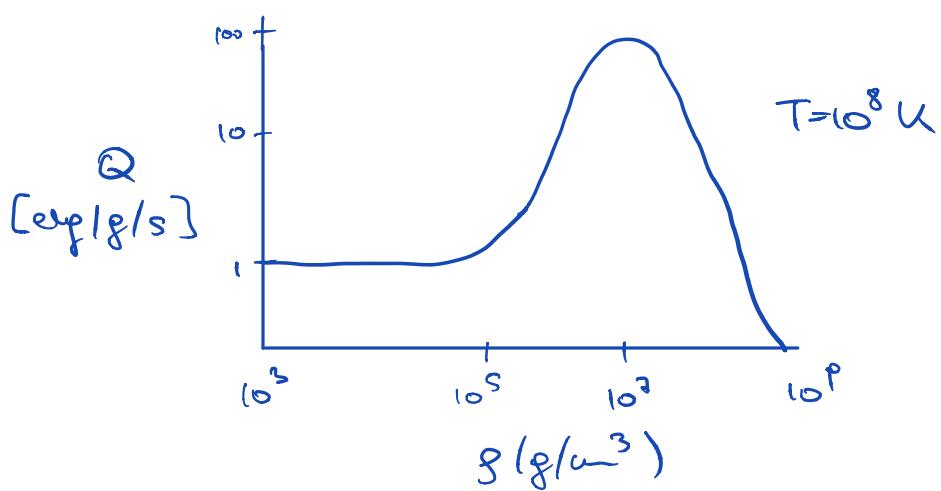
is understood.

With knowledge of  $\Gamma$ , one can compute the energy loss rate  $Q$

$$Q_T = \frac{2}{2\pi^2} \int_0^{\infty} dk k^2 \frac{\Gamma_T \omega}{e^{\omega/T} - 1} \quad \leftarrow \text{Box Emission disk of plasmas}$$

$$Q_L = \frac{1}{2\pi^2} \int_0^{k_1} dk k^2 \frac{\Gamma_L \omega}{e^{\omega/T} - 1}$$

for  $k > k_1$ , always all  
Kin. fallible



(representative for a  
"Heirarchical Bead's star or  
low-mass red-giant before  
He-ignition)

Anomalous energy loss (say from neutrino millideg e<sub>v</sub>)  
is constrained to be  $< 10 \text{ erg/g/s}$

$$\Rightarrow \text{e.g. } e_v \lesssim 10^{-14} \text{ e}$$

ST-energy loss impacts e.g. White-Dwarf cooling rate

