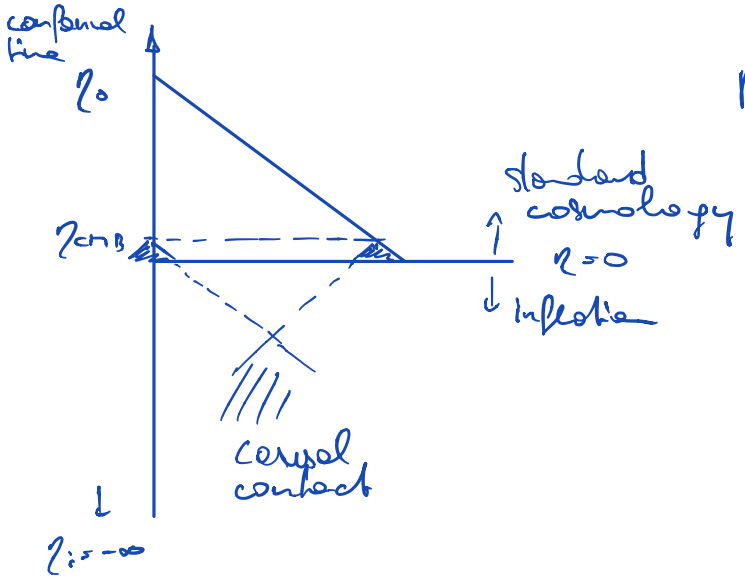
11. Quantum origin of structure

recall the idea of inflation



particle horizon

$$\chi_H(t) = \int_{t_i}^{t} \frac{dt'}{a(t')} = \eta(t) - \eta(t_i)$$

$$= \int_{a_i}^{a_0} \frac{1}{aH}\, d\ln a$$

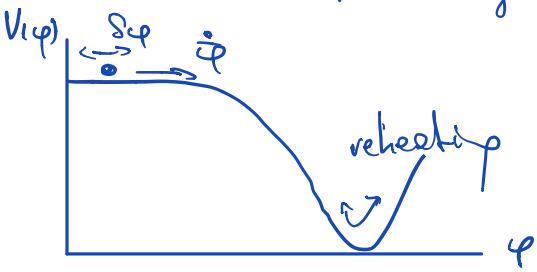
elapsed conformal time

depends on comoving Hubble radius $(aH)^{-1} \propto a^{1/2(1+3w)}$

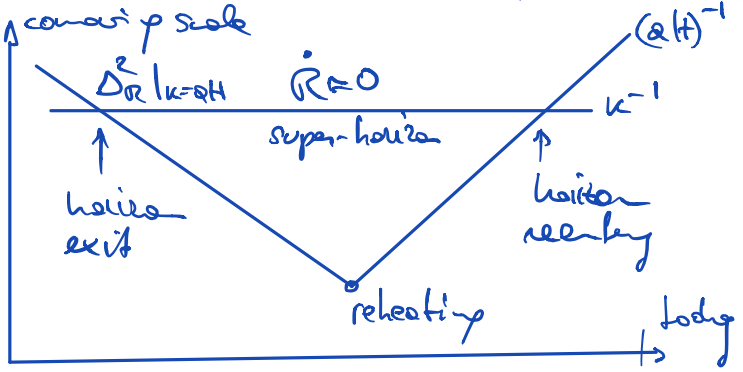
$$\eta \propto \frac{2}{1+3w}\, a_i^{1/2(1+3w)}$$

$\to -\infty$ for $w < -\frac{1}{3}$ and $a_i \to 0$

- inflation was originally invented to solve horizon (and flatness) problem.
- quantum fluctuations of scalar field that drives inflation gives seeds of structure



- strategy: compute R_k at horizon exit and make use of constancy of R at superhorizon scales



Canonical quantisation of harmonic osc.:

action of h.o. $S = \int dt \left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega(t)^2 x^2 \right) = \int dt\, L$

($m = 1$ for convenience; mass on a spring $\omega = k/m$)

e.o.m: $\ddot{x} + \omega^2 x = 0$

for quantizing the system, one first defines the canonical conjugate momentum to x

$$\pi = \frac{dL}{d\dot{x}} = \dot{x} \qquad \text{[here it coincides with } p = m\dot{x}\text{]}$$

$\Rightarrow$ elevate x, p to ops: $\underline{[\hat{x}, \hat{\pi}] = i\hbar}$

We are in the Heisenberg picture, for which $\hat{x} = \hat{x}(t)$

$$[\hat{x}(t), \dot{\hat{x}}(t)] = i\hbar$$

$\hat{x}$ is expanded in creation & annihilation ops, $\hat{a}$ & $\hat{a}^\dagger$

$$\underline{\hat{x}(t) = v(t)\,\hat{a} + v^*(t)\,\hat{a}^\dagger} \qquad v \ldots \text{complex mode fct.}$$

$\Rightarrow$ eom $\begin{cases} \ddot{v}(t) + \omega^2(t)\, v(t) = 0 \\ \ddot{v}^*(t) + \omega^2(t)\, v^*(t) = 0 \end{cases}$

$$[\hat{x}, \hat{p}] = (v\hat{a} + v^*\hat{a}^\dagger)(\dot{v}\hat{a} + \dot{v}^*\hat{a}^\dagger) - (\dot{v}\hat{a} + \dot{v}^*\hat{a}^\dagger)(v\hat{a} + v^*\hat{a}^\dagger)$$

$$= v\dot{v}^*\hat{a}\hat{a}^\dagger + v^*\dot{v}\hat{a}^\dagger\hat{a} - \dot{v}v^*\hat{a}\hat{a}^\dagger - \dot{v}^*v\,\hat{a}^\dagger\hat{a} + \text{terms that cancel}$$

$$= \underbrace{(v\partial_t v^* - (\partial_t v)\, v^*)}_{\text{purely imaginary; normalization can be chosen such that} = i}\, [\hat{a}, \hat{a}^\dagger] = i\hbar$$

$$\Rightarrow \underline{[\hat{a}, \hat{a}^\dagger] = 1} \qquad \text{with } \langle v, v\rangle \overset{!}{=} 1$$

$$\text{where } \langle v, w\rangle = \frac{i}{\hbar}\left[v^* \partial_t w - (\partial_t v^*) w\right]$$

Hilbert space built via

$$\begin{cases} \underline{\hat{a}\,|0\rangle = 0} & \text{defines vacuum} \\ |n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle & ; \text{ eigenstate of number op } \hat{N} = \hat{a}^\dagger \hat{a} \\ & \hat{N}|n\rangle = n|n\rangle \end{cases}$$

$$\begin{cases} \hat{a} = \langle v, \hat{x}\rangle = \frac{i}{\hbar}\Big[v^* (\partial_t v) \hat{a} + v^* (\partial_t v^*) \hat{a}^\dagger \\ \qquad - (\partial_t v) v \hat{a} - (\partial_t v^*) v^* \hat{a}^\dagger \Big] \\ \qquad = \frac{i}{\hbar}\left[v^* \partial_t v - (\partial_t v^*) v \right] \hat{a} = \langle v, v\rangle \hat{a} = \hat{a} \\ \hat{a}^\dagger = -\langle v^*, \hat{x}\rangle \end{cases}$$

Choice of mode fcts v that solves $x(t)$ is per se not unique, but will simply lead to a different set of $\hat{a}$ & $\hat{a}^\dagger$ via $\hat{a} = \langle v, \hat{x}\rangle$ ...

For $\omega(t) = \omega = \text{const}$ the preferred choice of v is the one that makes $|0\rangle$ the ground state of the Hamiltonian $\hat{H}$

$$\hat{H} = \dot{x}\pi - L$$

$$= \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\omega^2 \hat{x}^2$$

$$= \frac{1}{2}\left[(\dot{v}\hat{a} + \dot{v}^* \hat{a}^\dagger)(\dot{v}\hat{a} + \dot{v}^* \hat{a}^\dagger) + \omega^2 (v\hat{a} + v^* \hat{a}^\dagger)(v\hat{a} + v^* \hat{a}^\dagger) \right]$$

$$= \frac{1}{2}\left[(\dot{v}^2 + \omega^2 v^2)\hat{a}\hat{a} + (|\dot{v}|^2 + \omega^2 |v|^2)\hat{a}\hat{a}^\dagger + \text{c.c.} \right]$$

act on $|0\rangle$ with $\hat{a}|0\rangle = 0$

$$\hat{H}|0\rangle = \frac{1}{2}\left(\dot{v}^2 + \omega^2 v^2\right)\hat{a}^\dagger\hat{a}^\dagger + \left(|\dot{v}|^2 + \omega^2|v|^2\right)|0\rangle$$

if $|0\rangle$ is to be an eigenstate of $\hat{H}$, first term needs to vanish

$$\Rightarrow \quad \dot{v} = \pm i\omega v$$

$$\Rightarrow \langle v, v\rangle = \mp 2\omega|v|^2 \quad ; \text{ for } \langle v, v\rangle > 0 \text{ choose - sign}$$

$$\Rightarrow v(t) = \sqrt{\frac{\hbar}{2\omega}}\, e^{-i\omega t} \quad , \langle v, v\rangle = 1$$

$$\Rightarrow \hat{H} = \hbar\omega\left(\hat{N} + \tfrac{1}{2}\right) \; ; \text{ vacuum has minimum energy } \hbar\omega/2$$

Quantum fluctuations around the ground state ($\langle\hat{x}\rangle = 0$)

consider $\langle|\hat{x}|^2\rangle \equiv \langle 0|\hat{x}^\dagger\hat{x}|0\rangle$

$$= \langle 0|\left(v^*\hat{a}^\dagger + v\hat{a}\right)\left(v\hat{a} + v^*\hat{a}^\dagger\right)|0\rangle$$

$$= |v|^2\langle 0|\hat{a}\hat{a}^\dagger|0\rangle$$

$$= |v|^2\langle 0|[\hat{a}, \hat{a}^\dagger]|0\rangle = |v|^2$$

$$\Rightarrow \underline{\langle|\hat{x}|^2\rangle = |v|^2 = \frac{\hbar}{2\omega}}$$

quantum fluctuations of harmonic oscillator

Quantum fluctuations during inflation

Consider an inflationary period that is driven by scalar field φ:

$$S_\varphi = \int d^4x\sqrt{-g}\left[-\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) - V(\varphi)\right]$$

units $M_p = \frac{1}{8\pi G} = 1$.

A brute force way would be e.g. take Newtonian gauge with scalar perturb. ϕ and ψ & perturbed inflaton field $\varphi = \bar{\varphi}(\eta) + \delta\varphi(\vec{x},\eta)$ and work through the eqs. However, it turns out that the system can be reduced to 1 dynamical variable [e.g. absence of anisotropic stress in scalar theory implies $\phi = \psi$ and further constraint eq. reduces $\delta\varphi$ & ϕ to 1 variable].

An alternative way to treat inflationary perturbations to choose the following gauge (see Maldacena astro-ph/0210603)

$$\delta\varphi = 0 \qquad g_{ij} = -a^2\left[(1-2R)\,\delta_{ij} + h_{ij}\right]$$

comoving curvature perturbation from before

$$\partial_i h_{ij} = 0 \qquad h^i{}_i = 0 \qquad \text{tensor perturbation}$$

Only one metric perturbation $R(t,\vec{x})$ to follow, with the important property that it is constant on superhorizon-scales.

For quantization, we need the action S in second order in the perturbations. This is laborious, and here we simply quote the result

$$S = \frac{1}{2}\int d^4x\; a^3\,\frac{\dot{\bar{\varphi}}^2}{H^2}\left[\dot{R}^2 - \frac{1}{a^2}(\partial_i R)^2\right] + \text{higher order}$$

The action can be brought into canonical form by defining so-called "Mukhanov variables"

$$v \equiv z R \quad \text{with} \quad z \equiv a \frac{\dot\varphi}{H} = a \frac{\bar\varphi'}{\mathcal{H}} \qquad (\partial_\eta A = A')$$

[recall $\frac{dt}{d\eta} = a \quad \dot A = \frac{1}{a} A'$

$\mathcal{H} = \frac{a'}{a} = aH$]

$\Rightarrow$ in terms of conformal time

$$S = \frac{1}{2} \int d\eta\, d^3x \left[(v')^2 + (\partial_i v)^2 + \frac{z''}{z} v^2 \right] + \dots$$

= action of a harmonic oscillator with a time-dep. mass

$$m^2(\eta) = -\frac{z''}{z} \approx -\frac{2}{\eta^2} \quad \text{(de Sitter)}$$

$\Rightarrow$ eom in terms of Fourier-components $v_k(\eta) = \int d^3x\, e^{-i\vec k \cdot \vec x}\, v(\eta, \vec x)$

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0 \qquad \text{"Mukhanov-Sasaki-eqn."} \qquad k = |\vec k|$$

$$\omega_k^2(\eta) = k^2 - \frac{z''}{z}$$

classical evolution:

subhorizon $k^2 \gg |z''/z|$: $v_k'' + k^2 v_k = 0$

$\Rightarrow v_k \propto e^{\pm ik\eta}$ oscillation

superhorizon $k^2 \ll |z''/z|$: $\frac{v_k''}{v_k} = \frac{z''}{z} \approx \frac{2}{\eta^2}$

$\Rightarrow v_k \propto z \propto \frac{1}{\eta}$ [growing solution; η runs from negative values to 0 during inflation; decaying solution $v_k \propto \eta^2$]

$\Rightarrow \mathcal{R}_k = \frac{1}{z} v_k \approx \text{const}$ perturbation freezes at superhorizon scales

Canonical quantization of v:

- canonical conj. momentum $\pi = v'$
- Hamiltonian becomes $\hat{H}(\eta) = \frac{1}{2}\int d^3\vec{x}\left[\hat{\pi}^2 + (\partial_i \hat{v})^2 + m^2(\eta)\hat{v}^2\right]$
- Equal-time commutation relation:

$$\left[\hat{v}(\tau,\vec{x}), \hat{\pi}(\tau,\vec{x})\right] = i\delta^{(3)}(\vec{x}-\vec{y})$$

$$\hat{v}(\vec{x},\eta) = \int \frac{d^3k}{(2\pi)^3}\left[v_k(\eta)\hat{a}_k e^{i\vec{k}\cdot\vec{x}} + v_k^*(\eta)\hat{a}_k^\dagger e^{-i\vec{k}\cdot\vec{x}}\right]$$

$\Rightarrow \left[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}\right] = (2\pi)^3\delta(\vec{k}-\vec{k}')$; others 0

for normalization of $v_k(\eta)$ as $\langle v_k, v_k\rangle = 1$

one of the two boundary conditions of Mukhanov-Sasaki Eq (2nd order ODE)

- Hilbert-space is built by defining the vacuum state

$$\hat{a}_{\vec{k}}|0\rangle = 0$$

and repeated application of creation ops

$$|n_{\vec{k}_1}, n_{\vec{k}_2}, \ldots\rangle = \frac{1}{\sqrt{n!n!\ldots}}\hat{a}^\dagger_{\vec{k}_1}\hat{a}^\dagger_{\vec{k}_2}\ldots|0\rangle$$

of once mode-fcts v are selected, vacuum unambiguous;
however, $\langle v, v \rangle = 1$ not sufficient.

To see this, consider $u_k(\eta) = \alpha_k v_k(\eta) + \beta_k v_k^*(\eta)$
with $|\alpha_k|^2 - |\beta_k|^2 = 1$

$\Rightarrow \langle u, u \rangle = 1$ normalization unaffected and

$$\hat{v}(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left[\hat{b}_{\vec{k}} u_k(\eta) e^{i\vec{k}\cdot\vec{x}} + \hat{b}^\dagger_{\vec{k}} u_k^*(\eta) e^{-i\vec{k}\cdot\vec{x}} \right]$$

where b's satisfy equivalent commutation relations as a's;
both are related by so called "Bogoljubov transformations"

$$\hat{a}_{\vec{k}} = \alpha_k^* \hat{b}_{\vec{k}} + \beta_k \hat{b}^\dagger_{-\vec{k}} \quad \text{and h.c.}$$

and both can be used to construct Hilbert space via $\hat{a}_k |0\rangle = 0$ or
$\hat{b}_k |0\rangle = 0$

$\Rightarrow$ "b-vacuum" contains a-particles

$${}_b\langle 0 | \hat{N}_k^{(a)} | 0 \rangle_b = \underbrace{|\beta_k|^2}_{\text{density of } a\text{-particles}} \underbrace{\delta(0)}_{\text{volume factor}}$$

Physical vacuum:

for the harmonic osc. at the beginning it was the condition that $\hat{H}$
is minimized $\Rightarrow v_k(t) = \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k t}$ $\omega_k = \text{const}$

Now we face $\omega_k = \omega_k(\eta)$, so using the above solution implies $|0\rangle_{\eta_1} \neq |0\rangle_{\eta_0}$ and $|0\rangle_{\eta_0}$ is not the lowest energy state at a later time η_1.

Explicit time dep of $\hat{H}(\eta)$: energy of harmonic osc. is not conserved but is interacting with the gravitational field (this can lead to "gravitational prod." of particles where the energy is supplied by the classical grav. field)

Zero-point fluctuations in de Sitter phase

$$\omega_k^2(\eta) = k^2 - \frac{z''}{z} \simeq k^2 - \frac{a''}{a} = k^2 - \frac{2}{\eta^2}$$

at early times, all modes were subhorizon ($|k\eta| \ll 1$)

$\Rightarrow \omega_k^2(\eta) \simeq k^2 - \frac{2}{\eta^2} \to k^2$ = const, time-indep. freqn.

$\Rightarrow$ eom: $v_k'' + k^2 v_k = 0$

$\Rightarrow$ harmonic. osc. solution minimizing $\hat{H}$: $v_k = \frac{1}{\sqrt{2k}} e^{-ik\eta}$

$\Rightarrow$ Mukhanov Sasaki eqn is solved with initial cond.

$$\lim_{\eta \to -\infty} v_k(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta} \Rightarrow \text{"Bunch-Davies vacuum"}$$

the solution to $v_k'' + \left(k^2 - \frac{2}{\eta^2}\right) v_k = 0$ obeying this initial condition is

$$v_k(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) .$$

On superhorizon scales

$$\lim_{k\eta \to 0} v_k(\eta) = \frac{1}{i\sqrt{2}} \frac{1}{k^{3/2}\eta}$$

To compute the power-spectrum as a fct. of k, instead of $v(\vec{x},\eta) \to \hat{v}(\vec{x},\eta)$ we may promote the Fourier-components to operators $v_k \to \hat{v}_{\vec{k}} = v_k(\eta)\hat{a}_{\vec{k}} + v^*_{-k}(\eta)\hat{a}^\dagger_{-\vec{k}}$

(the a's fulfill the same commutation relations as before)

$$\langle 0|\hat{v}_{\vec{k}}\hat{v}_{\vec{k}'}|0\rangle = \langle 0|\left(v_k(\eta)\hat{a}_{\vec{k}} + v^*_{-k}(\eta)\hat{a}^\dagger_{-\vec{k}}\right)\left(v_k(\eta)\hat{a}_{\vec{k}'} + v^*_{-k'}(\eta)\hat{a}^\dagger_{-\vec{k}'}\right)|0\rangle$$

$$= v_k v^*_{-k'}\langle 0|\hat{a}_{\vec{k}}\hat{a}^\dagger_{-\vec{k}'}|0\rangle = v_k v^*_{-k'}\langle 0|[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{-\vec{k}'}]|0\rangle$$

$$= (2\pi)^3 |v_k|^2 \delta^{(3)}(\vec{k}+\vec{k}')$$

$$= (2\pi)^3 P_v(k)\, \delta^{(3)}(\vec{k}+\vec{k}')$$

$$\Rightarrow P_v(k) = |v_k|^2 \to \frac{1}{2k^3}\frac{1}{\eta^2} = \frac{\mathcal{H}^2}{2k^3} = \frac{(aH)^2}{2k^3} \quad \text{(superhorizon)}$$

or, in terms of the dim-less quantity $\Delta^2(k) \equiv \frac{k^3}{2\pi^2}P(k)$

$$\Delta^2_v = a^2\left(\frac{H}{2\pi}\right)^2$$

Now we are in a position to compute the power-spectrum of R at "horizon crossing" at $\eta = \eta_*$, $aH|_{\eta_*} = k$ after which it is conserved

$$R = \frac{H}{\dot{\varphi}}\frac{v}{a} \Rightarrow \Delta^2_R(k) = \left(\frac{H}{2\pi}\right)^2\left(\frac{H}{\dot{\varphi}}\right)^2\Bigg|_{k=aH}$$

$$\text{or } P_R(k) = \frac{1}{2k^3}\frac{H^4}{\dot\varphi^2}\Big|_{k=aH}$$

Different k-modes exit the horizon at slightly different times when aH has a different value; for slow roll inflation this procedure gives the correct result; in other cases one needs to numerically integrate Mukhanov-Sasaki eqs.

Slow-roll inflation:

recall from earlier lectures $\dot H = -\frac{1}{2}\dot\varphi^2$ (using Friedmann & eom) $4\pi G = \frac{1}{2}$

$$\Rightarrow \Delta_R^2 = \left(\frac{H}{2\pi}\right)^2\left(-\frac{1}{2}\frac{H^2}{\dot H}\right)\Big|_{k=aH}$$

we had the slow roll parameters $\varepsilon \equiv -\frac{\dot H}{H^2} \ll 1$ ($\ll 1$ for accelerated exp.) $\quad \eta \equiv \frac{\dot\varepsilon}{H\varepsilon} \ll 1$

$$\Rightarrow \Delta_R^2(k) = \frac{1}{8\pi^2}\frac{H^2}{\varepsilon}\Big|_{k=aH}$$

scale invariant spectrum $\propto k^0$ up to corrections from the time dependence of H & ε

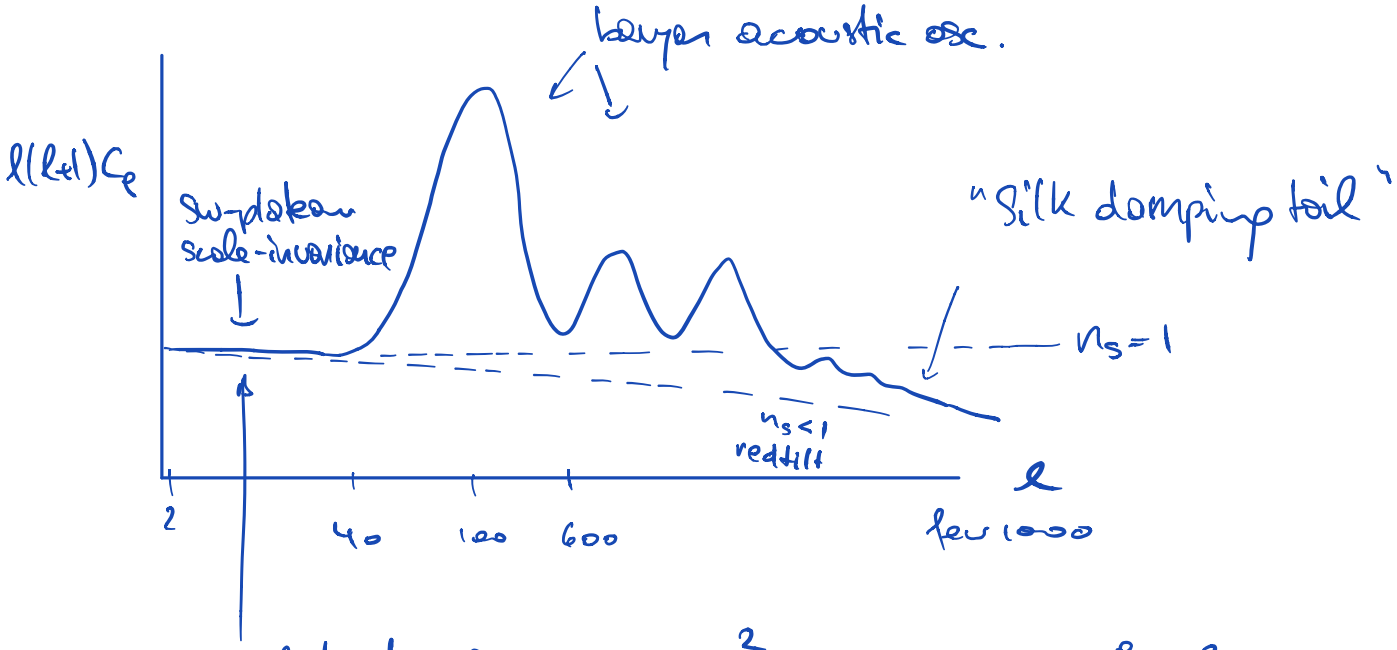
deviations from scale invariance measured by so-called "spectral index" n_s ($n_s = 1$ ... scale inv.)

$$n_s - 1 \equiv \frac{d\ln\Delta_R^2}{d\ln k} \xrightarrow[\text{roll}]{\text{slow}} -2\varepsilon - \eta$$

computable, once V is specified

Simple models of inflation predict $n_s < 1$ ("red tilted")

CMB measurements:



$\Rightarrow$ amplitude of spectrum $\Delta^2_R = 2{,}19(6) \cdot 10^{-9}$ } Planck

$n_s = 0{,}968 \pm 0{,}006$ }

Tensor perturbations h_{ij}:

expanding the Einstein Hilbert action in h_{ij} yields

$$S = \frac{m_P^2}{8} \int d\eta\, d^3x\, a^2 \left[(h'_{ij})^2 - (\partial_\ell h_{ij})^2 \right] + \text{higher order}$$

(we have re-introduced m_P to make h_{ij} dimensionless)

Fourier-expansion: (grav. wave has two polarizations)

$$h_{ij} = \int \frac{d^3k}{(2\pi)^3} \sum_{s=+,\times} \varepsilon^s_{ij}(k)\, h^s_{\vec{k}}(\eta)\, e^{i\vec{k}\cdot\vec{x}}$$

$$\varepsilon_{ii} = 0 \; ; \quad k^i \varepsilon_{ij} = 0 \qquad \varepsilon^s_{ij}\, \varepsilon^{s'}_{ij} = 2\delta_{ss'}$$

to get canonically normalized fields $v^s_{\vec{k}} \equiv \frac{a}{2} m_P h^s_{\vec{k}}$

$$\Rightarrow S = \sum_{s=+,\times} \frac{1}{2} \int d\eta\, d^3k \left[(v_k^{s\prime})^2 - \underbrace{\left(k^2 - \frac{a''}{a}\right)}_{\omega_k^2(\eta)} (v_k^s)^2 \right]$$

de Sitter: $\frac{a''}{a} = \frac{2}{\eta^2}$

$\Rightarrow$ precisely what we had for scalar perturbations
(two copies of it for the 2 pol.)

$$P_0 = \frac{1}{2k^3} (aH)^2 \Big|_{k=aH}$$

$\Rightarrow$ tensor power spectrum

$$P_t = P_h \cdot 2 = \frac{4}{k^3} \frac{H^2}{\pi_p^2} \Big|_{aH=k} \quad \text{or} \quad \Delta_t^2(k) = \frac{2}{\pi^2} \frac{H^2}{\pi_p^2} \Big|_{k=aH}$$

(2 pols)

observable: "tensor-to-scalar ratio" r (via CMB "B-modes")

$$r \equiv \frac{\Delta_t^2(k)}{\Delta_R^2(k)}$$

$\Delta_t^2(k) \leftarrow \propto H^2 \sim V_{\text{infl}}$

$\Delta_R^2(k) \leftarrow$ fixed by obs

slow-roll: $r = |16\varepsilon|_{k=aH}$

$\Rightarrow$ direct measure of the energy scale of inflation

$$V^{1/4} \sim \left(\frac{r}{0{,}01}\right)^{1/4} \cdot 10^{16}\,\text{GeV}$$

(future CMB-obs will probe to $r \sim 10^{-3}$)

Literature:

- Weinberg "Cosmology"
  everything calculated, but notation different from everyone else; great book

- Mukhanov "Physical Foundations of Cosmology"
  focus on derivations; very comprehensive

- Mukhanov "Quantum effects in gravity"
  more on last chapter and related topics

- Baumann lectures on inflation (arXiv)
  very pedagogical; parts taken from Mukhanov
  last lecture taken from there; other lectures inspired from there as well

- Dodelson "Modern Cosmology"
  very pedagogical; works with distribution fcts for photons
  ⇒ heavily aimed towards CMB