

## II. STANDARD COSMOLOGY + SHORT COMINGS

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JOSEF PRADLER

### 1. Some observational facts about our Universe:

- expanding, homogeneous & isotropic on scales larger than 100 Mpc  $\Rightarrow$  no vantage point is special

Typical length scales

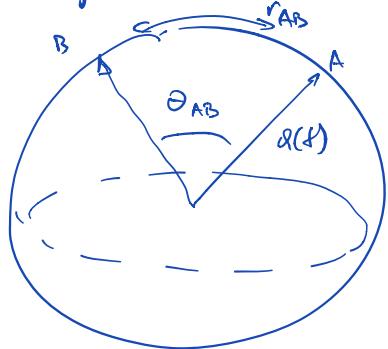
characteristic size of stars :	$1\text{ pc} \approx 3 \cdot 10^{18} \text{ cm} = 3,3 \text{ ly}$
distance to Galactic center (Andromeda) :	8 kpc (50 kpc)
characteristic size between galaxies :	17 pc
total observable patch :	several G pc

- filled with photons, with close to perfect blackbody spectrum  $T_b = 2,7255(6) \text{ K}$ , called cosmic microwave bkg. (CMB)
  - abundances of light elements (D & He) abnormally high, not originally from stars
    - points to Hubble Bang model
  - failure of predictions of Newton's laws that are based on visible components of matter; evidence for new forms of matter, called dark matter (DM);
    - points to new physics
  - Universe is expanding at accelerated pace (cosmological const. or some form of vacuum energy)
    - points to new physics
- $\Rightarrow$  all aspects are unified in the so-called standard cosmological model (ΛCDM), the basis of which we are going to describe now.

## 2. Hubble's law:

observational fact, that (sufficiently separated) galaxies recede from us with a velocity in proportion to their distance; proportionality constant is called "Hubble constant" where "constant" refers to consistency in space, but not necessarily in time.

consider expanding sphere with radius  $\alpha(t)$  & points A, B separated by  $\theta_{AB}$



$$r_{AB}(t) = \alpha(t) \theta_{AB}$$

⇒ relative velocity between A & B

$$\dot{r}_{AB} = \dot{\alpha}(t) \theta_{AB} = \frac{\dot{\alpha}}{\alpha} r_{AB}$$

$$H(t) = \frac{\dot{\alpha}}{\alpha} \quad \dots \text{Hubble parameter}$$

⇒ After sep. of variables and integration

$$r_{AB}(t) = \alpha(t) X_{AB}$$

↑

$$\text{with } \alpha(t) = e^{\int H(t) dt}$$

↑

analog of  $\theta_{AB}$ , called the comoving distance "scale factor"  
(integration constant)

$$H(t_0) = H_0 \approx 70 \text{ km/sec/Mpc} \quad [v_{pec} \approx 0(100 \text{ km/sec}) \text{ of large galaxies}]$$

$t_0 = 13.7 \text{ Gyr}$  age of the Universe;

measured by  $\frac{v}{d}$  of objects with small peculiar velocities

[or via Doppler shifts; d is hard, requires "standard candles" or "standard rulers"]

### 3. Robertson-Walker (RW) space-time

homogeneous & isotropic spaces are have largest symmetry, 3 rot + 3 boosties  
 3 possibilities: flat, 3D-sphere of constant positive curvature,  
 3D-hyperbolic space of const. negative curvature

for visualization, consider 2 spatial dimensions

e.g. 2D Sphere with radius  $\alpha$ , embedded in 3D Euclidean space w/  $x, y, z$ .

$$x^2 + y^2 + z^2 = \alpha^2 \Rightarrow dz = -\frac{x dx + y dy}{z} = \pm \frac{x dx + y dy}{\sqrt{\alpha^2 - x^2 - y^2}}$$

$$\Rightarrow \text{in 3D Euclidean metric } ds^2 = dx^2 + dy^2 + dz^2 = dx^2 + dy^2 + \frac{(x dx + y dy)^2}{\alpha^2 - x^2 - y^2}$$

$$ds, \text{ in polar coord. } (x = r \cos \varphi, y = r \sin \varphi) \quad ds^2 = \frac{dr^2}{1 - r^2/\alpha^2} + r^2 d\varphi^2$$

$\alpha \rightarrow \infty$  ... flat plane  
 $\alpha \rightarrow \text{imag.}$  ... hyperbolic space of const. negative curvature

rescaling of radial variable  $r = r' / \sqrt{1/\alpha^2}$

$$\Rightarrow ds^2 = \alpha^2 \left( \frac{dr^2}{1 - r^2} + r^2 d\varphi^2 \right)$$

$\begin{cases} K=1 & \text{--- space of const. positive curvature} \\ K=-1 & \text{--- space of const. negat. curvature} \\ K=0 & \text{--- flat space} \end{cases}$

$\Rightarrow$  generalise to 4D space with  $ds^2 = dt^2 - ds_{3D}^2$

$$ds^2 = dt^2 - \alpha^2(t) \left\{ \frac{dr^2}{1 - r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right\} \quad \text{FRW metric.}$$

$r, \theta, \varphi$  ... comoving coordinates. Observer at rest will remain so.

$t$  ... proper time measured by comoving ds.

(proof see below)

$$\left\{ g_{00} = +1 ; g_{rr} = -\frac{\alpha^2(t)}{1-\kappa r^2} ; g_{\theta\theta} = -\alpha^2(t) r^2 ; g_{\phi\phi} = -\alpha^2(t) r^2 \right.$$

elliptic form :

$$\text{write } dX^2 = \frac{dr^2}{1-\kappa r^2}$$

$$\Rightarrow X = \begin{cases} \sin^{-1} r & \kappa = -1 & X \in [0, \infty) \\ r & \kappa = 0 & X \in [0, \infty) \\ \sin^{-1} r & \kappa = +1 & X \in [0, \pi] \end{cases}$$

$$ds^2 = dt^2 - \alpha^2(t) \left\{ dX^2 + \begin{pmatrix} \sin^2 X \\ X^2 \\ \sin^2 X \end{pmatrix} (d\theta^2 + \sin^2 \theta d\varphi^2) \right\} \quad \begin{matrix} \kappa = -1 \\ \kappa = 0 \\ \kappa = +1 \end{matrix}$$

finally, it is often convenient to write the metric in  
"conformal time"  $\eta$

$$ds^2 = \alpha^2(\eta) [d\eta^2 - dl^2] \quad d\eta = \frac{dt}{\alpha(t)}$$

( flat FRW with  $\kappa=0$  is conformal to Minkowski, i.e. ds is equal to Minkowski times a conformal factor )

#### 4. Kinematics of FRW :

Christoffel-symbols for  $\underbrace{g_{ij}}_{=g_{ij} dx^i dx^j}$

$$ds^2 = dt^2 - \alpha^2(t) \left[ \frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right]$$

$$\left\{ \begin{array}{l} \Gamma_{00}^0 = 0 ; \quad \Gamma_{i0}^0 = 0 ; \quad \Gamma_{00}^i = 0 ; \\ \Gamma_{ij}^0 = \frac{1}{2} \left[ -\frac{\partial g_{ij}}{\partial t} \right] = \alpha^2 H g_{ij} ; \\ \Gamma_{j0}^i = \frac{-1}{2\alpha^2} g^{ik} \left[ \frac{\partial g_{jk}}{\partial t} \right] = +H S_j^i ; \\ \Gamma_{jk}^i = +\frac{1}{2} g^{ie} \left[ \frac{\partial g_{je}}{\partial x^k} + \frac{\partial g_{ke}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^e} \right] \end{array} \right.$$

If comoving observer at rest remains at rest

$$\text{geod. eqn. for } \vec{x}^r(t) : \frac{d^2 \vec{x}^r}{dt^2} + \Gamma_{\alpha\beta}^r \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0$$

$$\text{since } \Gamma_{\alpha\alpha}^i = 0 \Rightarrow \frac{d^2 \vec{x}}{dt^2} = 0$$

of Significance of the scale factor  $a(t)$ :

- proper distance at time  $t$  from origin to a comoving object at coord. distance  $x$

$$d(x, t) = \int_0^x \sqrt{g_{xx}} dx' = a(t) \int_0^x \frac{dx}{\sqrt{1 - \kappa x^2}} = a(t) \begin{cases} \sin^{-1} x & \kappa = +1 \\ x & \kappa = 0 \\ \sin^{-1} x & \kappa = -1 \end{cases}$$

$\Rightarrow$  proper distance between two comoving obs increases or decreases with  $a(t)$

$$\Rightarrow \text{rate of change } \dot{d} = d \frac{da}{a} = dH$$

- consider smoothed out vector field  $\vec{j}^r$

isotropy:  $\langle j^i \rangle = 0$ , i.e. no preferred direction

homogeneity:  $j^0(\vec{x}, t) = n(t)$  e.g. No. of galaxies, baryons

if  $j^r$  is a conserved quantity: etc. per proper volume in comoving free

$$0 = \nabla_r j^r = \partial_r j^r + \Gamma_{r\alpha}^r j^\alpha = \dot{n} + \Gamma_{;0}^r n = \dot{n} + 3Hn$$

$$\Rightarrow n(t) = \frac{\text{const}}{a^3(t)} \quad (\text{dilution by expanding proper Volume})$$

- vector field  $t_{ijr}$  that is compatible with homogeneity & isotropy on large scales

$$\text{isotropy: } \langle t_{ijr} \rangle|_{x=0} \propto \delta_{ij} (= g_{ij})$$

homogeneity: propagation velocity  $v$  is only a function of time

general covariance: spatial coord. basis that preserves the form of  $g_{ij}$ , while shifting the signs must not affect the propagation  $\langle t_{ijr} \rangle \propto g_{ij}$

$\Rightarrow$  Energy-momentum tensor must take everywhere the form

$$T^0_0 = g(t) \quad T^0_i = 0 \quad T^i_j = -\delta^i_j p(t)$$

$g(t), p(t)$  are the conventional definitions of proper energy density & pressure  $g(t) = \sum_i g_i$ ;  $p(t) = \sum_i p_i(t)$  ... all species

$\Rightarrow$  This is the Energy-momentum tensor of a perfect fluid.

$$\underline{T^r_s = (g + p) u^r u_s - p \delta^r_s}$$

$u^r$  — four-velocity of the fluid.  $u^r = dx^r/ds$

In the cosmic rest frame  $u^0 = 1$ ,  $u^i = 0$  and

$$(T^r_s) = \begin{pmatrix} g & & & \\ & -p & & \\ & & -p & \\ 0 & & 0 & -p \end{pmatrix}$$

Trace :  $T^r_r = g - 3p$

$\Rightarrow$  consider the implications of  $\nabla_r T^r_s = 0$  (for consistency)

$$\underline{\underline{\partial_r T^r_0 + \Gamma^r_{rs} T^s_0 - \Gamma^s_{r0} T^r_s = 0}}$$

$$\partial_t T^0_0 + \Gamma^i_{i0} T^0_0 - \Gamma^i_{i0} T^i_j = 0$$

$$\dot{g} + 3Hg - H\delta^i_j \delta^j_i (-p) = 0$$

$$\underline{\underline{\dot{g} + 3H(g+p) = 0}}$$

Solution for equation of state  $p = wg \Rightarrow g \propto a^{-3-3w}$

of "contents" of the universe

- Cold Matter (e.g. dust, non-relat. part)  $p \leq 0$

$$g \propto a^{-3}$$

- Hat Matter (radiation, relt.-part.)  $\rho = g/3$   

$$g \propto a^{-4}$$
- vacuum energy  $\rho = -g$   

$$g = \text{const.}$$

↳ Propagation of light, horizons, redshift

What fraction of the Universe is in causal contact?

observer at  $r_0, \theta_0, \varphi_0$  and we may take  $r_0=0$  (homogeneity)

labeled light ray that starts at  $t=0$  at position  $r_H$

$$\underline{ds^2 = 0} \quad (\text{light ray})$$

geodesics passing through  $r_0$  all lie on of constant  $\theta, \varphi$  ( $d\varphi = d\theta = 0$ )

$$ds^2 = 0 \Rightarrow dt^2 = a^2(t) \frac{dr^2}{1-kr^2} \Rightarrow \int_0^t \frac{dt'}{a(t')} = \int_0^{r_H} \frac{dr}{\sqrt{1-kr^2}}$$

recall proper distance at time  $t$

$$d_H(t) = d(r_H, t) = \int_0^{r_H} \sqrt{g_{tt}} dr$$

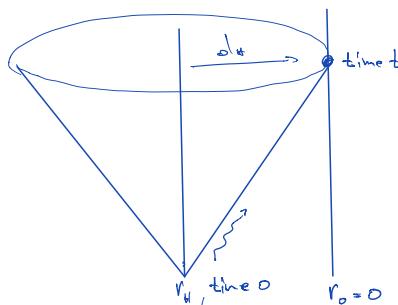
"particle horizon"

$$= a(t) \underbrace{\int_0^{r_H} \frac{dr}{\sqrt{1-kr^2}}}_{\text{comoving distance}} = a(t) \underbrace{\int_0^t \frac{dt'}{a(t')}}_{\text{proper time}}$$

⇒ if observable patch is finite

depends on behavior of  $a(t)$  for  $t \rightarrow 0$

⇒ we need to know the dynamics of  $a(t) \Rightarrow$  Einstein eqs.



Redshift: relate  $\frac{db}{\alpha(t)} = \frac{dr}{\sqrt{1-v^2}}$

time-indep.

$$\Rightarrow \frac{dt_1}{\alpha(t_1)} = \frac{dt_0}{\alpha(t_0)}$$

$dt_1, \dots$  interval of departure  
of two light signals  
 $dt_0 \dots$  interval of arrival

like as subsequent signals the wave crests

$$\Rightarrow \text{frequencies } \nu_1 = \frac{1}{dt_1}, \nu_2 = \frac{1}{dt_2}$$

$$\Rightarrow \frac{\nu_0}{\nu_1} = \frac{\alpha(t_1)}{\alpha(t_0)} \Rightarrow \alpha(t) \text{ increasing} \Rightarrow \text{red shift } z$$

$$1+z = \frac{\alpha(t_0)}{\alpha(t_1)}$$

### o) Distance measures

with close objects ( $z \lesssim 0,1$ ) spacetime curvature & effects of expansion on distance determinations can be neglected; they are used to measure  $H_0$ :

$$\begin{aligned} \alpha(t) &\approx \alpha(t_0) [1 + (t - t_0) H_0 + \dots] \\ \Rightarrow z &= H_0 \underbrace{(t_0 - t_1)}_{\approx d} + \dots \end{aligned}$$

(in units of  $c=1$ )

- geometric measures, such as parallax, to measure well only to  $d \lesssim 100 \text{ pc}$
- most common method is to use apparent luminosities  $\ell$  and compare it with the absolute luminosity of known objects  $L$

isotropic emission:  $\ell = \frac{L}{4\pi d^2}$   $[4\pi d^2 = \text{total energy passing through sphere}]$

(local)

most prominent example are "Cepheid variable" stars, very bright, luminosity time dep. ( pulsating ) ;  $\leftrightarrow$  relation between  $L$  & period  $\Rightarrow$  "standard candle" once  $L$  has been determined from nearby star.

Latest measurement  $H_0 = 73,24 \pm 1,74 \text{ km/sec/Mpc}$

$\Delta\chi^2/\nu = 1604.01424$  ( Riess et.al 2016)

[ NB: There is tension with Planck satellite meas.:  $69,3 \pm 0,7$  ]  
 $\left[ \text{cm}^{-3} \right]$

- at  $z > 0,1$  effects of cosmic expansion cannot be neglected  
apparent luminosity needs correction:

1. proper area of sphere around the luminous object at  $t_0$  (time of arrival)

$$4\pi r_i^2 \alpha^2(t_0) \quad r_i \dots \text{coor. dist. of coll.}$$

telescope has area  $\delta A$ ; fraction of sphere  $\delta A / 4\pi r_i^2 \alpha^2(t_0)$

$$\Rightarrow \text{in } L = \frac{L}{4\pi d_L^2} \text{ replace } \frac{1}{d^2} \rightarrow \frac{1}{r_i^2 \alpha^2(t_0)}$$

2. rate of arrival of individual photons is smaller by  $\frac{\alpha(t)}{\alpha(t_0)} = \frac{1}{1+z}$

3. energy of individual photons received ( $\hbar\nu_0$ ) smaller than emitted ( $\hbar\nu_i$ ) by a factor  $\frac{1}{1+z}$

$$L = \frac{L}{4\pi r_i^2 \alpha^2(t_0) (1+z)^2} = \frac{L}{4\pi d_L^2} \quad \dots \text{defines luminosity distance } d_L$$

$$\text{or } d_L = \alpha(t_0) r_i (1+z)$$

$$\textcircled{2} \text{ low redshift } z = H_0 (t_0 - t_i) + \frac{1}{2} (\varphi_0 + 2) H_0^2 (t_0 - t_i)^2 + \dots$$

$$\varphi_0 = - \frac{1}{H_0^2 \alpha(t_0)} \left. \frac{d^2 \alpha(t)}{dt^2} \right|_{t=t_0} \quad \text{"deceleration parameter"}$$

$$\text{EX: slow flow for } z \ll 1: \quad d_L \approx \frac{1}{H_0} \left[ z + \frac{1}{2} (1 - \varphi_0) z^2 + \dots \right]$$

$\Rightarrow$  measurement of  $H_0$  &  $\varphi_0$  by measuring  $d_L(z)$  to  $O(z^2)$  !

- for completeness, we mention another distance measure  
 "angular diameter distance"  $d_A$   
 consider Galaxy of proper diameter  $D$  at  $r_i$ , which emits light to  
 seen at present to subtend small angle  $\theta = \frac{D}{\alpha(t_i) r_i}$   $[D_{00} = \alpha^2 r^2]$   
 $d_A = \frac{D}{\theta} = \alpha(t_i) r_i$ ,  $\frac{da}{dt} = \frac{1}{(1+z)^2}$

Today we measure galaxies up to  $z \approx 7$ , & expansion of distance  
 measures in  $z$  does not make sense  $\Rightarrow$  we need full info on  
 the shape of  $\alpha(t)$ !  $\Rightarrow$  given by Einstein Eqs!

## 5. Cosmological solutions / Dynamics of Expansion

EX: derive from Einstein-Eqs

(i.e. explicitly calculate  $\Gamma_{RS}^{\mu\nu}$ ,  $R_{00}$ ,  $R_{ij}$ ,  $R$  for FRW)

$$\underline{0-0} \quad \frac{\ddot{\alpha}^2}{\alpha^2} + \frac{\kappa}{\alpha^2} = \frac{8\pi G}{3} g \quad \text{First Friedmann eqn.}$$

$$H(t) = \frac{\dot{\alpha}}{\alpha}$$

$$\underline{i-i} \quad 2\frac{\ddot{\alpha}}{\alpha} + \frac{\dot{\alpha}^2}{\alpha^2} + \frac{\kappa}{\alpha^2} = -8\pi G p \quad \text{Second Friedmann eqn.}$$

take difference

$$\underline{\underline{\frac{\ddot{\alpha}}{\alpha} = -\frac{4\pi G}{3} (g + 3p)}} \quad \text{Eqn. for acceleration-faster alone}$$

general features of Friedmann eqs.:

$\hookrightarrow$  if  $g > 0$ , the expansion can only stop if  $\kappa = +1$  (spherical)

2) we may define critical, present day energy density

$$\rho_c(t_0) = \frac{3H_0^2}{8\pi G} \approx 10^{-29} \text{ g/cm}^3$$

$$\Rightarrow k = \begin{cases} +1 & \text{if } \rho(t_0) > \rho_c(t_0) \\ 0 & \text{if } \rho(t_0) = \rho_c(t_0) \\ -1 & \text{if } \rho(t_0) < \rho_c(t_0) \end{cases}$$

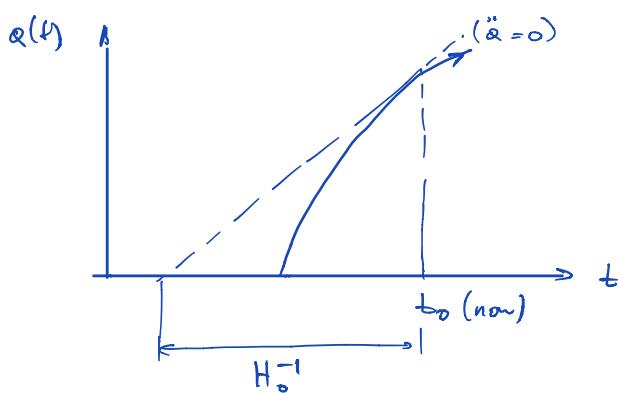
This statement is usually expressed in terms of a "density parameter"  $\Omega$

$$\Omega_0 = \frac{\rho(t_0)}{\rho_c(t_0)} \quad \Omega_0 = \sum_i \Omega_{0,i}$$

$$\Omega_0 = \begin{cases} > 1 & \Rightarrow k = +1 \\ = 1 & \\ < 1 & \Rightarrow k = -1 \end{cases}$$

- 3) if  $(\rho + 3p) > 0$  in the past (normal matter)  $\Rightarrow \ddot{\alpha} < 0$  deceleration,  
 $\dot{\alpha}/\alpha > 0$  always (a goes nonstaircally)  
 $\Rightarrow \alpha = 0$  was reached in past "Big Bang";

$\Rightarrow$  present age is less "Hubble time"  $t_0 < H_0^{-1}$



EX: compute cosmic time  $t$   
in various models (see below)

4) deceleration parameter from earlier can now be expressed as

$$q_0 = \frac{\rho(t_0) + 3p(t_0)}{2\rho_c(t_0)} = \frac{\Omega_0}{2} (1 + 3w)$$

5) compare  $g\dot{a}^2$  with  $K$  in the Friedmann equation

Rothkr.  $g \propto a^{-3} \Rightarrow g\dot{a}^2 \sim \frac{1}{a}$

Redfield  $g \propto a^{-4} \Rightarrow g\dot{a}^2 \sim \frac{1}{a^2}$

$\Rightarrow$  for sufficiently early times ( $a \rightarrow 0$ )  $K$  becomes negligible in

Friedmann eqn.  $H^2 \approx \frac{8\pi G}{3} g$ , i.e.  $g$  was very close to critical,  $g \approx g_c(t) = \frac{3H_0^2}{8\pi G}$ .

$\Rightarrow$  CMB observations show, that Universe today is still very close

to flat  $K=0$ ;  $|\frac{K}{a_0^2 H_0^2}| \ll 1$

EX: compute the spatial curvature parameter  ${}^{(3)}R$  in FRW

## Reminder from last lecture

FRW-metric: metric of a homogeneous, isotropic space

$$ds^2 = dt^2 - \alpha^2(t) \left\{ \frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right\} \quad K = \pm 1, 0$$

$$ds^2 = \alpha^2(\eta) [d\eta^2 - dl^2] \quad d\eta = \frac{dt}{\alpha(t)} \quad (\text{conformal form})$$

Symmetry imposes ideal fluid form of  $T_{\mu\nu}$

$$T^r_{\nu} = (g + p) u^r u_\nu - p \delta^r_\nu, \quad u^0 = 1, u^i = 0 \text{ in the cosmic r.f.}$$

$$\nabla_r T^r_{\nu} \Rightarrow \dot{g} + 3H(g+p) = 0$$

$$\text{with solution } p = w g \Rightarrow g \propto a^{-3-3w}$$

- Cold Rotker  $w=0, p=0, g \propto a^{-3}$

- Radiativer  $w=\frac{1}{3}, p=g/3, g \propto a^{-4}$

- vacuum energy  $w=-1, p=-g, g = \text{const.}$

Friedmann eqs:

$$\begin{cases} \frac{\dot{\alpha}^2}{\alpha^2} + \frac{K}{\alpha^2} = \frac{8\pi G}{3} g & H(t) = \frac{\dot{\alpha}}{\alpha} \\ \ddot{\alpha} = -\frac{4\pi G}{3} (g+3p) \end{cases}$$

we can define critical density  $\rho_c(t_0) = \frac{3H_0^2}{8\pi G} \approx 10^{-29} \text{ g/cm}^3$

and measure the various components in units of  $\rho_c$

$$\Omega_i = \frac{\rho_i(t_0)}{\rho_c(t_0)}$$

$$\Rightarrow \text{Friedmann eqn. } \Omega(t) - 1 = \frac{\kappa}{H^2 a^2} \quad \left\{ \begin{array}{l} > \\ = \\ < \end{array} \right\} 1 \Rightarrow \kappa = \begin{cases} +1 \\ 0 \\ -1 \end{cases}$$

Observationally (Planck):

$$\left\{ \begin{array}{l} \Omega_m = \Omega_b + \Omega_{cdm} \approx 5\% + 26\% \\ \Omega_\Lambda = 69\% \\ |\Omega_K| \lesssim 10^{-3} \\ \Omega_g \approx 5 \cdot 10^{-5} \text{ (CMB-photons)} \\ \Omega_{\text{gravity waves}} \lesssim 10^{-9} \end{array} \right. \quad \left\{ \begin{array}{l} \text{at \% - level} \\ \Lambda C D M - \text{standard cosmology} \end{array} \right.$$

$\kappa=0$  solutions to Friedmann eq:  $H^2 = \frac{8\pi G}{3} g$

non-dimension of  $a(t)$  enters only through  $\kappa/a^2(t)$  and for  $\kappa=0$  it has no significance  $\Rightarrow$  often, one takes  $a(t_0)=1$

- non-relat. matter dominated Universe

$$g(t) \approx g_n(t) = g_{n,0} \left( \frac{a_0}{a} \right)^3 \Rightarrow a(t) \propto t^{2/3} \Rightarrow H = \frac{2}{3t}$$

$$\left[ \frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3} g_{n,0}} \left( \frac{a_0}{a} \right)^{3/2} \Rightarrow da/\sqrt{a} = \text{const.} dt \Rightarrow t \propto a^{3/2} \right]$$

- relativistic, radiation-dominated Universe

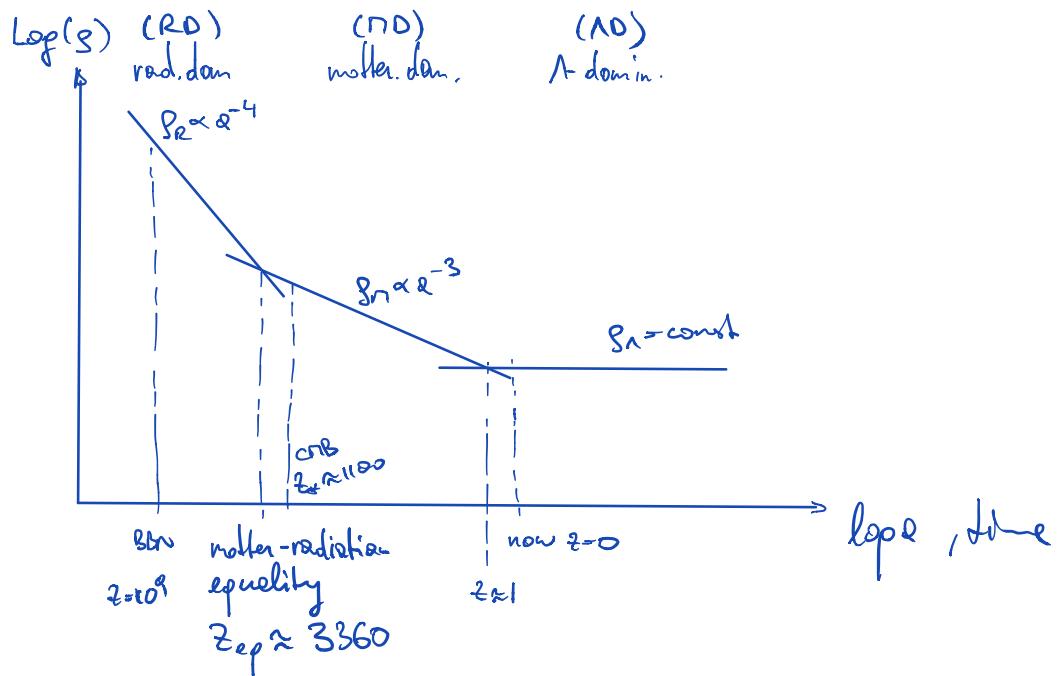
$$g(t) \approx g_r(t) = g_{r,0} \left( \frac{a_0}{a} \right)^4 \Rightarrow a(t) \propto \sqrt{t} \Rightarrow H = \frac{1}{2t}$$

- vacuum energy

$$g = \text{const} \Rightarrow a(t) \propto e^{Ht} ; H = \text{const.}$$

generally, one may write ( $\kappa=0$ )

$$\begin{aligned} H(t) &= H_0 \left[ \Omega_{n,0} \left( \frac{a_0}{a} \right)^3 + \Omega_{r,0} \left( \frac{a_0}{a} \right)^4 + \Omega_\Lambda \right]^{1/2} \\ &= H_0 \left[ \Omega_{n,0} (1+z)^3 + \Omega_{r,0} (1+z)^4 + \Omega_\Lambda \right]^{1/2} \end{aligned}$$



$$\begin{array}{lll}
 t=1\text{ sec} & t=380,000 \text{ yrs} & \text{today} \\
 T=1\text{ eV} & T=0.3\text{ eV} & t=13.8 \text{ Gyr} \\
 (10^0 \text{ K}) & (3000 \text{ K}) & T=2.7 \text{ K} \sim 10^{-5} \text{ eV}
 \end{array}$$

## Shortcomings in standard cosmology:

of "Flatness-problem"

$$\text{write Friedmann-eqn. in the form } \Omega(t) - 1 = \frac{K}{H^2 a^2}$$

$$|\Omega - 1| \text{ measures departure from flatness; } |\Omega_0 - 1| < 10^{-3} = \frac{K}{H_0^2 a_0^2} \text{ (Planck)}$$

$$\text{no: } H^2 = H_0^2 \Omega_0 \left(\frac{a_0}{a}\right)^3 \Rightarrow \frac{|\Omega - 1|}{|\Omega_0 - 1|} = \frac{H_0^2 a_0^2}{H^2 a^2}$$

$$\Rightarrow |\Omega - 1| < \frac{10^{-3}}{\Omega_0} \frac{a}{a_0} = \frac{10^{-3}}{\Omega_0} \frac{1}{(1+z)}$$

$$\text{at CNB } z=10^3 \Rightarrow |\Omega - 1| < 10^{-6}$$

and it gets worse when we carry the estimate further back to the RD epoch; if you carry it to the Planck time  $\leq 10^{-60}$  !

Horizon problem:

if we are going to study the causal structure of FRW spacetime;  
this is best done in conformal coord.; recall that

$$ds^2 = \alpha^2(\eta) [d\eta^2 - dX^2] \quad (\theta, \varphi = \text{const})$$

$$d\eta = \frac{dt}{\alpha(t)} \quad dX^2 = \frac{dt^2}{1-kx^2}$$

light geodesics from  $ds^2=0 \Rightarrow X(\eta) = \pm \eta + \text{const}$   
(i.e. 45°-lines in the  $X\eta$  plane)

• comoving distance a photon can travel from  $t_i$  to  $t$

$$\Delta X = \eta - \eta_i := \int_{t_i}^t \frac{dt'}{\alpha(t')}$$

[recall that the particle horizon considered earlier was  
 $d_{\text{ph}}(t) = \eta(t) \int_0^t \frac{dt'}{\alpha(t')}$  which is just the corresponding  
 proper distance]

$\Rightarrow$  for  $t_i = 0$  we get the comoving particle horizon

$$\underline{\underline{\eta}}(t) = \int_0^t \frac{dt'}{\alpha(t')} = \underline{\underline{\eta(t)} - \eta(0)}$$

• To elucidate the role of  $\eta$  note that we can rewrite

$$\eta = \int \frac{dt}{\alpha(t)} = \int \frac{1}{\alpha H} d\ln a$$

$\Rightarrow$  elapsed conformal time depends on the evolution of the  
comoving Hubble radius  $(\alpha H)^{-1}$

Consider a Universe that is dominated by one fluid with equation of state  $w$  (recall  $\rho = wS$ ,  $S \propto a^{-3-3w}$ )

$$(\rho H)^{-1} \propto a^{1/2(1+3w)}$$

$$\Rightarrow \eta \propto \frac{2}{1+3w} a^{1/2(1+3w)} \quad (\text{up to integration constant})$$

$\Rightarrow$  for conventional forms of matter ( $w > -\frac{1}{3}$ )  $\eta \propto a^{1/2(1+3w)} \rightarrow 0$  at the initial singularity at  $t_i=0$  where  $a(t)$  approaches 0.

$$\Rightarrow \text{for } t_i=0 \quad \eta(t_i) = \eta_i = 0$$

$\Rightarrow$  comoving particle horizon is finite  $X_h \propto a(t)^{1/2(1+3w)}$

(Figs. & much of this lecture from lecture notes of Benner)

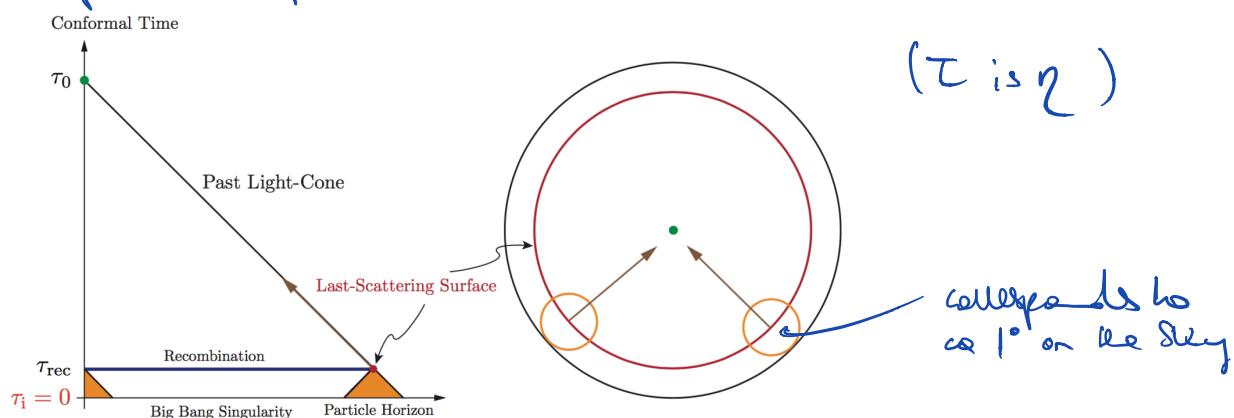


Figure 1.1: Conformal diagram for the standard FRW cosmology.

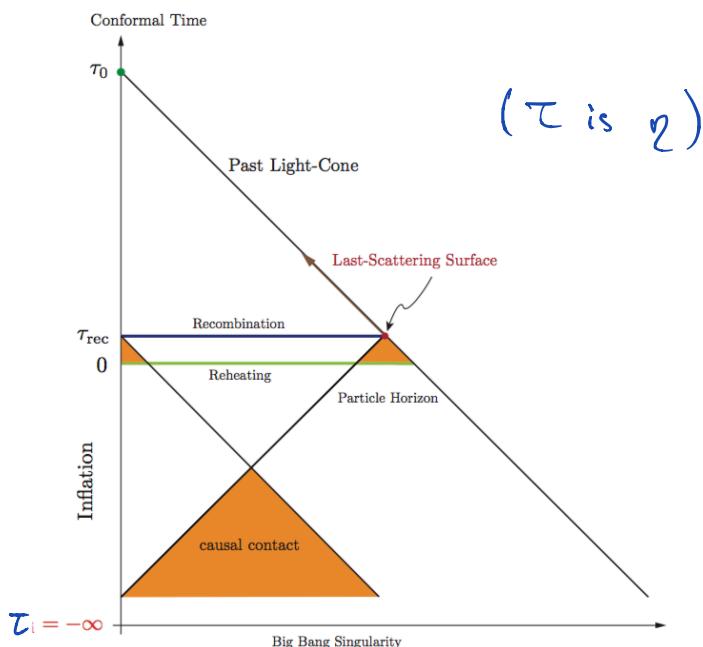
- non overlapping past light cones when CRB formed;  
i.e. regions were never in causal contact!
  - observationally  $\delta T/T|_{\text{CMB}} \sim 10^{-5}$
- $\Rightarrow$  Why is the CRB uniform to one part in  $10^8$  across the sky?

## Inflation

We saw that particle horizon is finite for  $\omega > -\frac{1}{3}$ . However, if we consider  $\omega < -\frac{1}{3}$

$$\eta(t_i) \propto \frac{2}{(1+3\omega)} a^{1/(1+3\omega)} \rightarrow -\infty$$

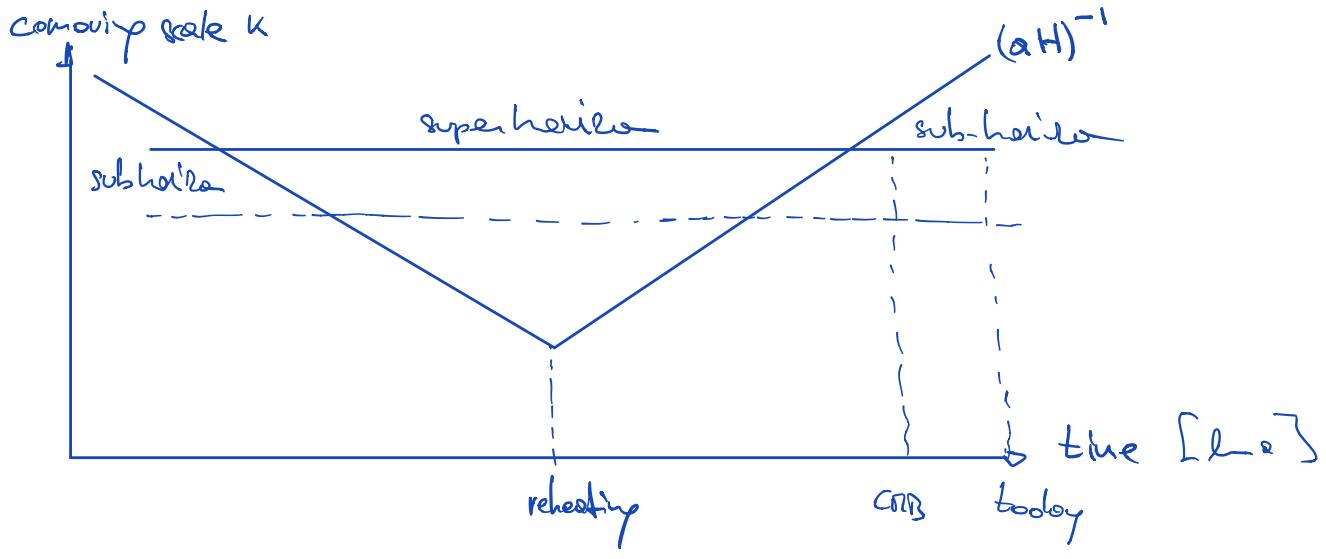
general condition is  $\frac{d}{dt} \left( \frac{1}{aH} \right) < 0 \Rightarrow$  comoving Hubble radius shrinks.



there was more conformal time between  $t_i=0$  & recombination than we had thought

$\eta = \int \frac{1}{aH} da$  is dominated by the lower boundary.

- ⇒ this (hypothetical) period is called inflation
- ⇒ two points that could have never been in causal contact in standard cosmology, can become so in inflationary Universe
- ⇒ the boundary is called "reheating"



"Inflationary phase"  $\leftrightarrow$  "standard cosmology"

Inflation implies:

Accelerated expansion:

$$\frac{d}{dt} \frac{1}{\alpha H} < 0 \quad \text{was required to solve horizon problem}$$

$$\frac{d}{dt} \frac{1}{\alpha} = - \frac{\ddot{\alpha}}{(\dot{\alpha})^2} < 0 \Rightarrow \ddot{\alpha} > 0 \Rightarrow \text{inflation implies accelerated expansion}$$

of slowly varying Hubble parameter

$$\frac{d}{dt} \frac{1}{\alpha H} < 0 \quad \Leftrightarrow \quad \varepsilon = - \frac{\dot{H}}{H^2} = - \frac{d \ln H}{dN} < 1 \quad dN = d \ln \alpha = H dt$$

$\varepsilon$  measures the time variation of the Hubble parameter  
(slow-roll parameter)

$dN$  measures the number of "e-folds"  $N$  of inflationary expansion

$\Rightarrow N = 40 - 60$  to solve the horizon problem

$\Rightarrow$  inflation must last long enough; measured by

$$\eta = \frac{\dot{\varepsilon}}{H\varepsilon} = \frac{d \ln \varepsilon}{dN}$$

for  $|\eta| < 1$ , fractional change of  $\varepsilon$  per Hubble time is small  
& inflation persists

of negative pressure ( $\omega < -\frac{1}{3}$ )

## Slow-roll inflation :

scalar field  $\phi$  ("inflaton"), minimally coupled to gravity

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{2} + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]$$

$(\nabla_\mu \phi = \partial_\mu \phi \text{ for scalar})$

Variation of  $\phi$  yields:

$$\nabla^\mu \left[ g^{\mu\nu} (\partial_\nu \phi) (\partial_\lambda \delta\phi) - \frac{\partial V}{\partial \phi} \delta\phi \right] = 0$$

$$\stackrel{\text{P.I.}}{=} \left[ -2\nu \left[ \nabla^\mu \left( g^{\mu\nu} \partial_\nu \phi \right) \right] - \nabla^\mu \frac{\partial V}{\partial \phi} \right] \delta\phi = 0$$

$$\Rightarrow \frac{1}{\sqrt{-g}} \partial_\nu \left[ \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \right] + \frac{\partial V}{\partial \phi} = 0$$

consider spatially homogeneous field  $\phi = \phi(t)$        $\partial_i \phi = 0$

$$g = -\frac{Q(t)^6 r^4}{1 - \kappa t^2} \quad \frac{1}{\sqrt{-g}} \partial_t \sqrt{-g} = \frac{1}{\sqrt{-g}} 3H \sqrt{-g}$$

$$\Rightarrow \text{eom } \ddot{\phi} + 3H\dot{\phi} + V' = 0 \quad V' = \frac{\partial V}{\partial \phi}$$

Energy momentum tensor:

$$[\partial^\mu \phi = g^{\mu\lambda} \partial_\lambda \phi]$$

$$T_{\nu}^{\mu} = \partial^\mu \phi \partial_\nu \phi - \left( \frac{1}{2} \partial^\mu \phi \partial_\nu \phi - V(\phi) \right) \delta^\mu_\nu$$

(eom would also follow from  $\nabla_\mu T^\mu_\nu = 0$ )

can be brought into perfect fluid form  $T^\mu_\nu = (\rho + p) u^\mu u_\nu - p \delta^\mu_\nu$

$$g = \frac{1}{2} (\partial^r \phi) (\partial_r \phi) + V(\phi) \quad p = \frac{1}{2} (\partial^r \phi) (\partial_r \phi) - V(\phi) \quad u^r = \frac{\partial^r \phi}{\sqrt{(\partial^r \phi) (\partial^r \phi)}}$$

for  $\dot{\phi} = \dot{\phi}(+)$

$$\underline{g = \frac{1}{2} \dot{\phi}^2 + V(\phi)} \quad p = \frac{1}{2} \dot{\phi}^2 - V(\phi)$$

(This is the velocity  
of energy current,  
see Weinberg (1972), App.)

$\Rightarrow \omega = \frac{p}{g}$  is in general time-dependent;  $\omega \geq -1$  for  $V > 0$

$$\text{if } \phi = \phi_0 = \text{const} \Rightarrow p = -g = -V(\phi_0); T_{rr} = V(\phi_0) g_{rr} \\ \Lambda = 8\pi G V(\phi_0)$$

$$\Rightarrow H^2 = \frac{8\pi G}{3} g = \frac{8\pi G}{3} \left( \frac{1}{2} \dot{\phi}^2 + V \right) \quad \text{minimizes cosm. const.}$$

$$\text{take time derivative } \frac{d}{dt} H^2 = 2H\dot{H} = \frac{8\pi G}{3} (\dot{\phi}\ddot{\phi} + V'\dot{\phi})$$

$$\stackrel{\text{eqn}}{=} -8\pi G H \dot{\phi}^2$$

$$\Rightarrow \dot{H} = -4\pi G \dot{\phi}^2$$

change in  $H$  due to kinetic energy of  $\phi$

if we want field inflation to work, the fractional change  $(\dot{H}/H)(1/H)$  in  $H$  during an expansion time  $1/H$  must be small

$$\Sigma = \frac{|\dot{H}|}{H^2} \ll 1 \Rightarrow \frac{3 \cdot 4\pi G \dot{\phi}^2}{8\pi G (\frac{1}{2} \dot{\phi}^2 + V)} \ll 1 \Leftrightarrow \dot{\phi}^2 \ll |V(\phi)|$$

$$\Rightarrow \text{during inflation } p \approx -g \quad H \approx \sqrt{\frac{8\pi G}{3} V(\phi)} \approx \text{const.}$$

• let us solve for  $a(\phi)$  ; assume  $\dot{\phi} \ll 3H\dot{\phi}$  (related to  $V$ )

$$H = \frac{da}{dt} = \dot{\phi} \frac{da}{d\phi} \underset{\text{can}}{\approx} - \frac{V'}{3H} \frac{da}{d\phi}$$

$$\Rightarrow -V' \frac{da}{d\phi} = 3H^2 = 8\pi G V$$

$$\Rightarrow \frac{a(\phi_2)}{a(\phi_1)} \underset{\text{---}}{\approx} e^{-8\pi G \int_{\phi_1}^{\phi_2} \frac{V}{V'} d\phi}$$

• from  $\mathcal{L} \ll 1 \Rightarrow \left| \frac{V'}{V} \right| \ll \sqrt{16\pi G}$

$\Rightarrow$  argument in the exponential above is much greater than  $\sqrt{16\pi G} |\phi_1 - \phi_2|$

$\Rightarrow$  we get a large number of e-foldings in any time-interval in which  $\phi$  changes by at least  $\Delta\phi \gtrsim \frac{1}{\sqrt{16\pi G}} \sim 3,4 \cdot 10^{18}$  GeV  
 $\Rightarrow$  nonplanckian field excursions are suggested.

Note: does not necessarily rule out classical treatment of gravity; condition to neglect quantum-gravity effects is that energy density  $\ll$  Planckian energy density

$$|V(\phi)| \ll \frac{1}{(4\pi G)^2}$$

This is possible by endowing  $V$  with a sufficiently small coupling constant  $g$ ; all arguments above are indep. of  $g$ .

• back to flatness problem:  $|1 - 1| = \frac{\kappa}{\dot{\phi}^2 H^2} \propto \frac{1}{\dot{\phi}^2} \sim \frac{1}{e^{2n}}$  in inflation ( $H \approx \text{const}$ )

$\Rightarrow$  Universe is driven to be very close to critical; solves the flatness prob.