Optimal Oil Exploration and Extraction with Delay

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Outline

1. History of exploration and extraction of non-renewable resources
2. Greiner/Semmler model of optimal extraction
3. Optimal control problems with delays
4. Model according to Liu and Sutinen
5. Maximum principle
6. Numerical results
7. Singular control
8. Delay in exploration
Market data

BP Statistical Review June 2009
EIA Performance Profiles of Major Energy Producers 2007

Oil price in years 1850 – 2008
Seminal papers on modelling

Hotelling 1936: no exploration, fixed stock, alternative market price rules, calculus of variations

Pindyck 1978: stock can be increased by exploration, extraction cost depend on size of stock, U–shaped price path, maximum principle

Liu, Sutinen 1982: formal justification of optimal exploration and extraction, singular control

Swierzbinski, Mendelsohn 1989: criticism on determination of extraction cost, price always rising

Greiner, Semmler 2008: extraction and feedback rule of exploration, some numerical solutions (Grüne)
Greiner/Semmler model: state and control variables

\( x_0 \) : total resource at time \( t = 0 \)
\( \tilde{x}_0 < x_0 \) : total attainable resource at time \( t = 0 \)
\( x^e(t) \) : explored resource at time \( t \in [0, T], \ T > 0 \) final time
\( y(t) \) : total extraction in \([0, t], \ y(0) = 0\)
\( x^s(t) \) : stock of resource, \( x^s(t) = x^e(t) - y(t) \)
\( u(t) \) : rate of extraction (control)

Dynamics with feedback rule of exploration:

\[
\frac{dx^e(t)}{dt} = \dot{x}^e(t) = \xi(\tilde{x}_0 - x^e(t)),
\]

\( \Rightarrow \ x^e(t) \to \tilde{x}_0 \) for \( t \to \infty \).

DATA: \( x_0 = 6, \ \tilde{x}_0 = 3, \ \xi = 0.5 \)
Greiner/Semmler model: state and control variables

- $x_0$: total resource at time $t = 0$
- $	ilde{x}_0 < x_0$: total attainable resource at time $t = 0$
- $x^e(t)$: explored resource at time $t \in [0, T], \ T > 0$ final time
- $y(t)$: total extraction in $[0, t]$, $y(0) = 0$
- $x^s(t)$: stock of resource, $x^s(t) = x^e(t) - y(t)$
- $u(t)$: rate of extraction (control)

Dynamics with feedback rule of exploration:

$$ \frac{dx^e(t)}{dt} = \dot{x}^e(t) = \xi (\tilde{x}_0 - x^e(t)),$$

$$ \Rightarrow x^e(t) \to \tilde{x}_0 \quad \text{for} \quad t \to \infty. $$

**DATA:** $x_0 = 6, \ \tilde{x}_0 = 3, \ \xi = 0.5$
Greiner/Semmler model: price and cost function

Price function (per unit)

\[ p(y, x^s, u) = \left( \frac{1}{\gamma + q_1 \cdot u + q_2 \cdot x^s - q_3 \cdot y} \right)^\alpha \]

DATA: \( \gamma = 0.05, \; q_1 = 1, \; 0 \leq q_2 \leq 0.5, \; 0 \leq q_3 \leq 0.1 \).

Cost of extraction (per unit)

\[ c(y) = \Phi \left( x_0 - y \right)^2 \]

DATA: \( \Phi = 2, \; x_0 = 6 \).
**Greiner/Semmler model: price and cost function**

**Price function (per unit)**

\[ p(y, x^s, u) = \left( \frac{1}{\gamma + q_1 \cdot u + q_2 \cdot x^s - q_3 \cdot y} \right)^{\alpha} \]

**DATA:** \( \gamma = 0.05, \ q_1 = 1, \ 0 \leq q_2 \leq 0.5, \ 0 \leq q_3 \leq 0.1. \)

**Cost of extraction (per unit)**

\[ c(y) = \frac{\Phi}{(x_0 - y)^2} \]

**DATA:** \( \Phi = 2, \ x_0 = 6. \)
Greiner/Semmler: Optimal Control Model

Maximize benefit

Determine an extraction rate $u$ that maximizes the benefit

$$J(x^s, y, u) = \int_0^T e^{-rt} (p(y, x^s, u) - c(y)) u \, dt$$

subject to

$$\dot{x}^s(t) = -u(t) + \xi(\tilde{x}_0 - x^s(t) - y(t)), \quad x^s(0) = x^e_0,$$

$$\dot{y}(t) = u(t), \quad y(0) = 0,$$

$$0 \leq u(t) \leq u_{\text{max}} = 0.1, \quad 0 \leq x^s(t) \quad \forall \ t \in [0, T].$$

Optimal control is continuous, since the strict Legendre condition holds.
Numerical results: $T = 50$, $q_1 = 1$, $q_2 = q_3 = 0$

Initial conditions $x^s(0) = 0.1, 1.5, 3.0$

stock $x^s(t)$ with $x(T) > 0$

extraction $u(t)$

accumulated extraction $y(t)$

price function $p(x^s(t), u(t))$
Numerical results: $T = 100$, $q_1 = 1$, $q_2 = q_3 = 0$

Initial conditions $x^s(0) = 0.1, 1.5, 3.0$

stock $x^s(t)$ with $x(T) = 0$

extraction $u(t)$

accumulated extraction $y(t)$

price function $p(x^s(t), u(t))$
Numerical results: $T = 150$, $q_1 = 1$, $q_2 = q_3 = 0$

Initial conditions $x^s(0) = 0.1, 1.5, 3.0$

stock $x^s(t) : x(t) = 0$, $t_e \leq t \leq T$

extraction $u(t)$

accumulated extraction $y(t)$

price function $p(x^s(t), u(t))$
Numerical results: \( T = 50, q_1 = 1, q_2 > 0, q_3 = 0 \)

**Price function (per unit)**

\[
p(y, x^s, u) = \left( \frac{1}{\gamma + q_1 \cdot u + q_2 \cdot x^s - q_3 \cdot y} \right)^\alpha
\]

**DATA:** \( \gamma = 0.05, q_1 = 1, 0 \leq q_2 \leq 0.5, 0 \leq q_3 \leq 0.1 \).

\( u(t) \) for \( q_2 = 0 \)

\( u(t) \) for \( q_2 = 0.005 \)
Numerical results: $T = 50$, $q_1 = 1$, $q_2 > 0$, $q_3 = 0$

- $u(t)$ for $q_2 = 0.005$
- $u(t)$ for $q_2 = 0.01$
- $u(t)$ for $q_2 = 0.1$
- $u(t)$ for $q_2 = 0.2$
Delays in developing resources

Dynamics of exploration with a delay $d > 0$:

$$\dot{x}^e(t) = \xi(\tilde{x}_0 - x^e(t - d)), \quad x^e(\tau) = x_0^e, \quad -d \leq \tau \leq 0,$$

Solution $x^e(t)$ becomes oscillatory for higher values of $d$.

Dynamics with a state delay $d > 0$

$$\dot{x}^s(t) = -u(t) + \xi(\tilde{x}_0 - x^s(t - d) - y(t - d)),
\dot{y}(t) = u(t), \quad y(\tau) = 0, \quad -d \leq \tau \leq 0,
0 \leq u(t) \leq u_{\text{max}} \quad \forall \ t \in [0, T].$$

Maximize the benefit

$$J(x^s, y, u) = \int_0^T e^{-rt}(p(y, x^s, u) - c(y)) \ u \ dt$$
Numerical results: $T = 50$, $q_1 = 1$, $q_2 = q_3 = 0$

Initial condition $x^s(0) = 0.1$ and delay $d = 2$

stock $x^s(t)$ with $x(T) > 0$

extraction $u(t)$

accumulated extraction $y(t)$

price function $p(x^s(t), u(t))$
Retarded Optimal Control Problem

State \( x(t) \in \mathbb{R}^n \), Control \( u(t) \in \mathbb{R}^m \)

Dynamics and Boundary Conditions

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t), x(t-r), u(t), u(t-s)), \quad \text{a.e. } t \in [0, T], \\
x(t) &= \varphi(t), \quad t \in [-r, 0) \quad \text{(state delay } r \geq 0 \text{)}, \\
u(t) &= \psi(t), \quad t \in [-s, 0) \quad \text{(control delay } s \geq 0 \text{)}, \\
b(x(T)) &= 0
\end{align*}
\]

Mixed control-state constraints or pure state constraints

\[
C(x(t), u(t)) \leq 0, \quad S(x(t)) \leq 0, \quad t \in [0, T]
\]

Minimize

\[
J(u, x) = g(x(T)) + \int_0^T f_0(t, x(t), x(t-r), u(t), u(t-s)) \, dt
\]
Retarded Optimal Control Problem

State \( x(t) \in \mathbb{R}^n \), Control \( u(t) \in \mathbb{R}^m \)

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\]
Retarded Optimal Control Problem

State $x(t) \in \mathbb{R}^n$, Control $u(t) \in \mathbb{R}^m$

Dynamics and Boundary Conditions

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t), x(t - r), u(t), u(t - s)), \text{ a.e. } t \in [0, T], \\
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Mixed control-state constraints or pure state constraints

\[
C(x(t), u(t)) \leq 0, \quad S(x(t)) \leq 0, \quad t \in [0, T]
\]

Minimize

\[
J(u, x) = g(x(T)) + \int_0^T f_0(t, x(t), x(t - r), u(t), u(t - s)) \, dt
\]
Discretization

For simplicity consider a MAYER-type problem with cost functional

\[ J(u, x) = g(x(T)) \]

Assume: there exists a stepsize \( h > 0 \) and integers \( k, l, N \in \mathbb{N} \) with

\[ r = k \cdot h, \quad s = l \cdot h, \quad T = N \cdot h. \]

**EULER discretization at grid points**

\[ t_i := ih, \quad i = 0, 1, \ldots, N. \]

**Approximation at grid points:**

\[ u(t_i) \approx u_i \in \mathbb{R}^m \quad (i = 0, \ldots, N-1), \quad x(t_i) \approx x_i \in \mathbb{R}^n \quad (i = 0, \ldots, N) \]
Minimize

\[ J(u, x) = g(x_N) \]

subject to

\[ x_{i+1} - x_i - h \cdot f(t_i, x_i, x_{i-k}, u_i, u_{i-1}) = 0, \quad i = 0, \ldots, N - 1, \]
\[ b(x_N) = 0, \]
\[ C(x_i, u_i) \leq 0, \quad i = 0, \ldots, N, \]
\[ S(x_i) \leq 0, \quad i = 0, \ldots, N, \]
\[ x_{-i} := \varphi(-ih) \quad (i = 0, \ldots, k), \quad u_{-i} := \psi(-ih) \quad (i = 1, \ldots, l). \]

Optimization Variable:

\[ z := (u_0, x_1, u_1, x_2, \ldots, u_{N-1}, x_N) \in \mathbb{R}^{N(m+n)} \]
Discretization and NLP–Solvers

- AMPL: Programming language (Fourer, Gay Kernighan)
- IPOPT: Interior point method (Wächter et al.)
- LOQO: Interior point method (Vanderbei et al.)
- Other NLP solvers embedded in AMPL: cf. NEOS server
- Special feature: solvers provide LAGRANGE-multipliers
A model according to Liu and Sutinen

State and control variables

\( x_0 \) : total resource at time \( t = 0 \), \( x_0 = 10 \),  
\( x_0^e \) : explored resource at time \( t = 0 \), \( x_0^e \in [0.1, 3] \),  
\( x^e(t) \) : explored resource at time \( t \in [0, T] \), \( T = 100 \) periods  
\( y(t) \) : total extraction in time interval \([0, t] \), \( y(0) = 0 \)  
\( x^s(t) \) : stock of resource, \( x^s(t) = x^e(t) - y(t) \), \( x^s(0) = x^e(0) \)  
\( w(t) \) : rate of exploration (control)  
\( u(t) \) : rate of extraction (control)

Dynamics

**Exploration**

\[
\frac{dx^e(t)}{dt} = \dot{x}^e(t) = w(t)
\]

**Extraction**

\[
\frac{dy(t)}{dt} = \dot{y}(t) = u(t)
\]
A model according to Liu and Sutinen

State and control variables

\[ x_0 \quad : \quad \text{total resource at time } t = 0, \ x_0 = 10, \]
\[ x_0^e \quad : \quad \text{explored resource at time } t = 0, \ x_0^e \in [0.1, 3], \]
\[ x^e(t) \quad : \quad \text{explored resource at time } t \in [0, T], \ T = 100 \text{ periods} \]
\[ y(t) \quad : \quad \text{total extraction in time interval } [0, t], \ y(0) = 0 \]
\[ x^s(t) \quad : \quad \text{stock of resource, } x^s(t) = x^e(t) - y(t), \ x^s(0) = x^e(0) \]
\[ w(t) \quad : \quad \text{rate of exploration (control)} \]
\[ u(t) \quad : \quad \text{rate of extraction (control)} \]

Dynamics

Exploration

\[ \frac{dx^e(t)}{dt} = \dot{x^e}(t) = w(t) \]

Extraction

\[ \frac{dy(t)}{dt} = \dot{y}(t) = u(t) \]
### Price and cost functions (per unit)

**Price function**

\[ p(y, u) = p_{\text{max}}(y) \left( 1 - \frac{u}{m} \right) \]

**Prohibitive price**

\[ p_{\text{max}}(y) = p_0 + \frac{p_1}{\varepsilon + x_0 - y} \]

**DATA:** \( m = 1, \ p_0 = 7, \ p_1 = 1, \ \varepsilon = 0.1. \)

**Cost of extraction**

\[ c(x_e, y) = c_0 + \frac{c_1}{\delta + x_e - y} \]

**Cost of exploration**

\[ k(x_e) = k_0 + \frac{k_1}{\xi + x_0 - x_e} \]

**DATA:** \( k_0 = 1, \ k_1 = 0, \ \xi = 1, \ c_0 = 1, \ c_1 = 4, \ \delta = 1. \)
Price and cost functions (per unit)

**Price function**

\[ p(y, u) = p_{\text{max}}(y) \left(1 - \frac{u}{m}\right) \]

**Prohibitive price**

\[ p_{\text{max}}(y) = p_0 + \frac{p_1}{\varepsilon + x_0 - y} \]

**DATA:** \( m = 1, \ p_0 = 7, \ p_1 = 1, \ \varepsilon = 0.1. \)

**Cost of extraction**

\[ c(x^e, y) = c_0 + \frac{c_1}{\delta + x^e - y} \]

**Cost of exploration**

\[ k(x^e) = k_0 + \frac{k_1}{\xi + x_0 - x^e} \]

**DATA:** \( k_0 = 1, \ k_1 = 0, \ \xi = 1, \ c_0 = 1, \ c_1 = 4, \ \delta = 1. \)
Optimal Control Model

Determine exploration rate $w(t)$ and extraction rate $u(t)$ that maximize the benefit

$$J(x^e, y, w, u) = \int_0^T e^{-rt}((p(y, u) - c(x^e, y))u - k(x^e)w))dt$$

subject to the dynamics and control and state constraints

$$\dot{x}^e(t) = w(t), \quad x^e(0) = x_0^e$$
$$\dot{y}(t) = u(t), \quad y(0) = 0$$
$$0 \leq w(t) \leq w_{\text{max}}, \quad 0 \leq u(t)$$
$$y(t) \leq x^e(t), \quad x^e(T) \leq x_0$$

DATA: $w_{\text{max}} = u_{\text{max}} = 0.5$. 

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Optimal Oil Exploration and Extraction with Delay
Optimal Control Model

Determine exploration rate $w(t)$ and extraction rate $u(t)$ that maximize the benefit

$$J(x^e, y, w, u) = \int_0^T e^{-rt}((p(y, u) - c(x^e, y))u - k(x^e)w)dt$$

subject to the dynamics and control and state constraints

$$\dot{x}^e(t) = w(t), \quad x^e(0) = x^e_0$$
$$\dot{y}(t) = u(t), \quad y(0) = 0$$
$$0 \leq w(t) \leq w_{\text{max}}, \quad 0 \leq u(t)$$
$$y(t) \leq x^e(t), \quad x^e(T) \leq x^0$$

DATA: $w_{\text{max}} = u_{\text{max}} = 0.5$. 
Maximum principle

Benefit function

\[ B(u, x^e, y) = (p(u, y) - c(x^e, y)) \, u \]

The following calculations are carried out for a general benefit.

Augmented Hamiltonian: normal case

\[
H(t, x^e, y, u, w, \lambda_{x^e}, \lambda_y) = e^{-rt} \left( B(u, x^e, y) - k(x^e)w \right) + \lambda_{x^e} \, w + \lambda_y \, u + \nu(x^e - y)
\]

\[ \lambda = (\lambda_{x^e}, \lambda_y) : \text{adjoint variable} \]

\[ \nu : \text{multiplier for state constraint } x^e - y \geq 0 \]

Switching function for control \( w \) appearing linearly

\[ \sigma_w = \lambda_{x^e} - ke^{-rt} \]
### Maximum principle

#### Condition for $u > 0$

$$0 = H_u = e^{-rt}B_u + \lambda_y$$

$$\Rightarrow u = u(x^e, y, \lambda_y)$$

#### Condition for $w$

$$w(t) = \begin{cases} 
  w_{\text{max}}, & \sigma_w(t) > 0 \\
  0, & \sigma_w(t) < 0 \\
  \text{singular}, & \sigma_w(t) = 0 
\end{cases}$$

#### Adjoint equations and transversality: there exist $\alpha, \gamma \geq 0$

$$\dot{\lambda}_{x^e} = -H_{x^e} = e^{-rt}(k_{x^e}w - B_{x^e}) - \nu, \quad \lambda_{x^e}(T) = \gamma - \alpha$$

$$\dot{\lambda}_y = -H_y = -e^{-rt}B_y + \nu, \quad \lambda_y(T) = -\gamma$$

$$\nu(x^e - y) = 0, \quad \gamma(x^e(T) - y(T)) = 0, \quad \alpha(x_0 - x^e(T)) = 0$$
$T = 100$, $x_0^e = 0.1$, $k_0 = 0.5$, $k_1 = 0$

extraction $u$, exploration $w$  

$w$ and switching function

$x^e$, $x^s$, $y$

price $p(u, y)$
$T = 100$, $x_0^e = 0.1$, $k_0 = 1.5$, $k_1 = 0$

extraction $u$, exploration $w$

$w$ and switching function

$x^e$, $x^s$, $y$

price $p(u, y)$
\( T = 100, \ x_0^e = 0.1, \ k_0 = 3, \ k_1 = 0 \)

extraction \( u \), exploration \( w \)

\( x^e, \ x^s, \ y \)
$T = 100$, $x_0^e = 4$, $k_0 = 1.5$, $k_1 = 0$

extraction $u$, exploration $w$  

$w$ and switching function

$x^e$, $x^s$, $y$

price $p(u, y)$
$T = 50, \ x^e_0 = 0.1, \ k_0 = 1, \ k_1 = 2$ (non-constant cost)

*extraction* $u$, *exploration* $w$

$\chi^e, \ \chi^s, \ y$
Optimal Control Structure: $x_0^e = 0.1$, $k_0 = 1.5$, $k_1 = 0$

**extraction** $u$, **exploration** $w$

- **boundary arc** $x^e(t) = y(t) = x_0 = 10$ and $u(t) = 0$
  for $t_e = 71.7 \leq t \leq T = 100$.

- **extraction control** $u$ is continuous.

- **exploration control** $w$ is bang–singular–bang.

  **singular arc** in the interval $[11.1, 32.2]$. 

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The extraction control satisfies $u(t) > 0$ on a singular arc. Hence, the relation $\lambda_y(t) = -B_u [t] e^{-rt}$ holds. Differentiation:

$$\dot{u} = \frac{B_y + r B_u - B_{uy} u - B_{ux} e^w}{B_{uu}}$$

**Singular arc:** the switching function and its derivative vanish

$$0 = \sigma_w = \lambda_{xe} - k e^{-rt},$$

$$0 = \dot{\sigma}_w = e^{-rt} (rk - B_{xe})$$
Singular control of order one

\[ 0 = \ddot{w} = e^{-rt}(rk_{xe} w - B_{xe} u \dot{u} - B_{xe}w - B_{xe} y u) \]
\[ = \mathcal{A}(t, u, x^e, y) + \mathcal{B}(t, u, x^e, y) w \]

Thus the singular control is of order \( q = 1 \).

\[ w = -\frac{\mathcal{A}}{\mathcal{B}} = \frac{(B_{uu}B_{xe}y - B_{xe}uB_{uy})u + B_{xe}u(B_y + rB_u)}{B_{uu}(rk_{xe} - B_{xe}x^e) + (B_{xe}u)^2} \]

- Strict Generalized Legendre Condition \( \mathcal{B}[t] > 0 \) holds.
- Values agree precisely with computed singular control.
$d > 0$: number of periods to explore new deposits.

Modify the objective functional by introducing the delay into the objective functional and the state constraint:

Maximize the benefit

$$J(x^e, y, w, u) = \int_0^T e^{-rt} \left[ (p(y(t), u(t)) - c(x^e(t - d), y(t)))u(t) - k(x^e(t))w(t) \right] dt$$

Dynamics and state constraint

$$\dot{x}^e(t) = w(t), \quad x^e(\tau) = x_0^e \quad \text{for} \quad -d \leq \tau \leq 0,$$

$$\dot{y}(t) = y(t), \quad y(0) = 0,$$

$$y(t) \leq x^e(t - d).$$
\[ T = 100, \ x_0^e = 0.1, \ k_0 = 1.5, \ k_1 = 0 : d = 0 \text{ and } d = 4 \]

**extraction** \( u \), **exploration** \( w \)

\[ d = 0 \]

\[ d = 4 \]
Numerical solutions for two optimal control models of oil exploration and extraction.

Model of Greiner/Semmler: optimal extraction and fixed rate of exploration.

Model of Liu/Sutinen: optimal extraction and exploration.

Exploration control: bang-bang or bang-singular-bang.

Delays in exploring new deposits: needs better understanding.